

UNIFICATION OF EXTREMAL LENGTH GEOMETRY ON TEICHMÜLLER SPACE VIA INTERSECTION NUMBER

HIDEKI MIYACHI

ABSTRACT. In this paper, we give a framework for the study of the extremal length geometry of Teichmüller space after S. Kerckhoff, F. Gardiner and H. Masur. There is a natural compactification using extremal length geometry introduced by Gardiner and Masur. The compactification is realized in a certain projective space. We develop the extremal length geometry in the cone which is defined as the inverse image of the compactification via the quotient mapping. The compactification is identified with a subset of the cone by taking an appropriate lift. The cone contains canonically the space of measured foliations in the boundary.

We first extend the geometric intersection number on the space of measured foliations to the cone, and observe that the restriction of the intersection number to Teichmüller space is represented explicitly by the formula in terms of the Gromov product with respect to the Teichmüller distance. From this observation, we deduce that the Gromov product extends continuously to the compactification.

As an application, we obtain an alternative approach to Earle-Ivanov-Kra-Markovic-Royden's characterization of isometries. Namely, with some few exceptions, the isometry group of Teichmüller space with respect to the Teichmüller distance is canonically isomorphic to the extended mapping class group. We also obtain a new realization of Teichmüller space, a hyperboloid model of Teichmüller space with respect to the Teichmüller distance.

CONTENTS

1. Introduction	2
1.1. Background	2
1.2. Motivation	3
1.3. Results	4
1.4. Plan of this paper	9
2. Teichmüller theory	10
2.1. Teichmüller space	11
2.2. Measured foliations	11
2.3. Extremal length	11
2.4. Kerckhoff's formula	12
3. The Gardiner-Masur closure	12
3.1. Function \mathcal{E}_p	12

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3.2. Properties of \mathcal{E}_p	13
4. Cones \mathcal{C}_{GM} , \mathcal{T}_{GM} and $\tilde{\partial}_{GM}$	14
4.1. Cones	14
4.2. Models of \mathcal{C}_{GM} , \mathcal{T}_{GM} and $\tilde{\partial}_{GM}$	14
5. Intersection number and Extremal length associated to a point	15
5.1. Intersection number associated to the basepoint	15
5.2. Extremal length on \mathcal{MC}_{GM} associated to the basepoint	16
5.3. Extremal length is intrinsic	17
6. Topology of \mathcal{MC}_{GM}	18
6.1. Bounded sets are precompact	18
6.2. A system of neighborhoods	19
7. The Gromov product and an extension of \mathcal{E}_ζ	20
7.1. The Gromov product for d_T	21
7.2. Equicontinuity	21
8. An extension of the intersection number	22
8.1. Extension of the intersection number i_{x_0}	23
8.2. Intersection number is intrinsic	25
8.3. Extension of the Gromov product	26
9. Isometric action of $\mathcal{T}_{g,m}$	26
9.1. Mapping of bounded distortion for triangles	26
9.2. Null space	27
9.3. ω preserves \mathcal{PMF}	29
9.4. Proof of Theorem 3	31
9.5. Proof of Corollary 2	31
10. Hyperboloid model	34
Acknowledgements	35
References	35

1. INTRODUCTION

1.1. **Background.** The *Teichmüller distance* is a canonical and important distance on Teichmüller space. The geometry of the Teichmüller distance is deeply related to the extremal length geometry on that space (cf. [28]).

To the author's knowledge, in [19], S. Kerckhoff first studied the boundary of Teichmüller space at infinity via extremal length, and the extremal length geometry on Teichmüller space was formulated precisely by F. Gardiner and H. Masur in [13]. Indeed, in [13], they defined a compactification of Teichmüller space, recently called the *Gardiner-Masur compactification*, which is defined by collecting the asymptotic behavior of projective classes of extremal lengths of simple closed curves (cf. §1.3). The definition of their compactification is similar to the Thurston compactification. In fact, the Thurston compactification of Teichmüller space is defined by collecting the asymptotic behavior of hyperbolic lengths of simple closed curves (cf. [7]). Actually, Gardiner and Masur observed in [13] a standard relation between the two boundaries. Namely, the Gardiner-Masur boundary contains the Thurston boundary in the sense that the space of projective measured foliations is contained in the Gardiner-Masur boundary.

By definition, any point in the Gardiner-Masur boundary is the projective class of a function on the set of homotopy classes of non-trivial and non-peripheral simple closed curves. In [33], the author observed that any boundary point is represented by a continuous function on the space of measured foliations, while any point of the Thurston boundary is represented by the intersection number function. Thus, the observation in [33] sets our expectations that the Gardiner-Masur boundary have properties which are similar to those in the rich theory of the Thurston boundary.

1.2. Motivation.

1.2.1. *Unification of Teichmüller geometry in terms of intersection number.* In the original definition of the Thurston compactification of Teichmüller space, we recognize each point of Teichmüller space as a function on the set of simple closed curves by assigning the hyperbolic lengths of simple closed geodesics. After the recognition, we take the closure of the set of projective classes of such functions in the projective space to get the compactification (cf. [7]). In a broad sense, completions due to Thurston carry out with recognizing each point of Teichmüller space as a function on the set of simple closed curves (see also [8]). The Gardiner-Masur compactification is defined by the same manner as the Thurston compactification by considering the square root of extremal length instead of the hyperbolic length (cf. (1.2)). Hence, the Gardiner-Masur compactification is considered as an object in the category “Thurston’s completion”. Thus, it is expected, as mentioned in the previous section, that every boundary point of the Gardiner-Masur compactification is recognized as the projective class of a function on the set of simple closed curves defined by (a kind of) intersection number.

In [5], F. Bonahon realized the Thurston compactification in the space of *geodesic currents*. Indeed, in his method, any point of Teichmüller space is associated to an equivariant Radon measure on the space of hyperbolic geodesics on the universal cover of the base surface of Teichmüller space. He extended the notion of intersection number function to the space of geodesic currents. It should be noted that he gave a unified treatment for the Thurston compactification in terms of the intersection number. His theory is broadly applied in many fields in mathematics and yields enormous rich results (cf. e.g. [4] and [8]).

Thus, it is natural to ask :

Question 1. *Can we develop extremal length geometry in terms of intersection number ?*

1.2.2. *Relation to the geometry on the Teichmüller distance.* As discussed in the previous section, the space \mathbb{R}_+^S of non-negative functions on the set of simple closed curves admits a distance is the ambient space of Thurston’s completion. The interior in the ambient space \mathbb{R}_+^S admits a distance

$$(1.1) \quad d_\infty(f, g) = \log \sup_{\alpha \in S} \left\{ \frac{f(\alpha)}{g(\alpha)}, \frac{g(\alpha)}{f(\alpha)} \right\}$$

which is perceived as the product distance of countably many 1-dimensional hyperbolic spaces. Possibly $d_\infty(f, g) = \infty$ for some $f, g \in \mathbb{R}_+^S$ and the topology from (1.1) is different from the product topology on \mathbb{R}_+^S .

From Kerckhoff’s formula (2.8), the Teichmüller distance is represented by the ratios of extremal lengths. Moreover, a natural lift (1.3) of the Gardiner-Masur embedding gives an isometric embedding from Teichmüller space to the ambient

space $(\mathbb{R}_+^S, d_\infty)$. Hence, it is quite natural to expect that any geometric property of Teichmüller distance has an effect on the geometric structure of the Gardiner-Masur compactification, and vice versa.

Question 2. *How is the geometry of Teichmüller distance related to the geometry of the Gardiner-Masur compactification (embedding) ?*

1.3. Results. The purpose of this paper is to develop the extremal length geometry on Teichmüller space after Kerckhoff, Gardiner and Masur. Indeed, aiming for a counterpart for Bonahon's theory, we attempt to unify the extremal length geometry via intersection number.

1.3.1. Notation. We fix the notation to give our results precisely. Let $X = X_{g,m}$ be a Riemann surface of genus g with m punctures such that $2g - 2 + m > 0$. Denote by $\mathcal{T}_{g,m}$ the Teichmüller space of X . It is known that the geometric structure of the Teichmüller space is independent of the choice of basepoints. Especially, the statement of our Theorem 1 below does not depend on the basepoint. When the argument depends on the basepoint, we consider the Teichmüller space $\mathcal{T}_{g,m}$ as a pointed space $(\mathcal{T}_{g,m}, x_0)$, where $x_0 = (X, id)$.

Let \mathcal{S} be the set of non-peripheral and non-trivial simple closed curves on X , and \mathcal{MF} the space of measured foliations. By definition, the space \mathcal{MF} is contained in the space $\mathbb{R}_+^{\mathcal{S}}$ of non-negative functions on \mathcal{S} (cf. §2.2).

In [13], Gardiner and Masur proved that a mapping

$$(1.2) \quad \Phi_{GM}: \mathcal{T}_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \text{Ext}_y(\alpha)^{1/2}] \in \mathbb{P}\mathbb{R}_+^{\mathcal{S}}$$

is an embedding and the image is relatively compact, where $\text{Ext}_y(\alpha)$ is the extremal length of $\alpha \in \mathcal{S}$ on $y \in \mathcal{T}_{g,m}$. The closure $\text{cl}_{GM}(\mathcal{T}_{g,m})$ of the image is called the *Gardiner-Masur closure* or *compactification*, and the complement of the image in the closure is said to be the *Gardiner-Masur boundary*, which we denote by $\partial_{GM}\mathcal{T}_{g,m}$. Thus, the Gardiner-Masur compactification $\text{cl}_{GM}(\mathcal{T}_{g,m})$ is realized in the projective space $\mathbb{P}\mathbb{R}_+^{\mathcal{S}}$ of $\mathbb{R}_+^{\mathcal{S}}$.

We consider the cone \mathcal{C}_{GM} which is defined as the inverse image of $\text{cl}_{GM}(\mathcal{T}_{g,m})$ via the projection $\mathbb{R}_+^{\mathcal{S}} \rightarrow \mathbb{P}\mathbb{R}_+^{\mathcal{S}}$ (cf. §4.1). Notice that $\mathcal{MF} \subset \mathcal{C}_{GM}$ because $\partial_{GM}\mathcal{T}_{g,m}$ contains the space \mathcal{PMF} of projective measured foliations as noted in §1.1. One of our aims in this paper is to define the intersection number function on \mathcal{C}_{GM} . In order to avoid any confusion, we denote by $I(\cdot, \cdot)$ the original geometric intersection number function on \mathcal{MF} .

1.3.2. Unification by intersection number. Notice that the Gardiner-Masur embedding (1.2) admits a natural lift

$$(1.3) \quad \tilde{\Phi}_{GM}: \mathcal{T}_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \text{Ext}_y(\alpha)^{1/2}] \in \mathcal{C}_{GM} \subset \mathbb{R}_+^{\mathcal{S}}.$$

Our unification is stated as follows.

Theorem 1 (Unification). *There is a unique continuous function*

$$i(\cdot, \cdot): \mathcal{C}_{GM} \times \mathcal{C}_{GM} \rightarrow \mathbb{R}$$

with the following properties.

- (i) *For any $y \in \mathcal{T}_{g,m}$, the projective class of the function $\mathcal{S} \ni \alpha \mapsto i(\tilde{\Phi}_{GM}(y), \alpha)$ is exactly the image of y under the Gardiner-Masur embedding. Actually, it holds*

$$i(\tilde{\Phi}_{GM}(y), \alpha) = \text{Ext}_y(\alpha)^{1/2}$$

for all $\alpha \in \mathcal{S}$.

- (ii) For $\mathbf{a}, \mathbf{b} \in \mathcal{C}_{GM}$, $i(\mathbf{a}, \mathbf{b}) = i(\mathbf{b}, \mathbf{a})$.
- (iii) For $\mathbf{a}, \mathbf{b} \in \mathcal{C}_{GM}$ and $t, s \geq 0$, $i(t\mathbf{a}, s\mathbf{b}) = ts i(\mathbf{a}, \mathbf{b})$.
- (iv) For any $y, z \in \mathcal{T}_{g,m}$,

$$i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(z)) = \exp(d_T(y, z)).$$

In particular, we have $i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(y)) = 1$ for $y \in \mathcal{T}_{g,m}$.

- (v) For $F, G \in \mathcal{MF} \subset \mathcal{C}_{GM}$, the value $i(F, G)$ is equal to the geometric intersection number $I(F, G)$.

1.3.3. Unification by intersection number with basepoint. According to the technical reason, instead of Theorem 1, we really prove the following *basepoint-dependent* version which is a paraphrase of Theorem 1 above (cf. §1.4).

Theorem 2 (Unification with basepoint). *Fix a basepoint $x_0 \in \mathcal{T}_{g,m}$. There is an embedding $\Psi_{x_0}: \mathcal{T}_{g,m} \rightarrow \mathcal{C}_{GM}$ and a unique continuous function*

$$i(\cdot, \cdot): \mathcal{C}_{GM} \times \mathcal{C}_{GM} \rightarrow \mathbb{R}$$

independent of the choice of basepoint with the following properties.

- (i) For any $y \in \mathcal{T}_{g,m}$, the projective class of the function $\mathcal{S} \ni \alpha \mapsto i(\Psi_{x_0}(y), \alpha)$ is exactly the image of y under the Gardiner-Masur embedding.
- (ii) For $\mathbf{a}, \mathbf{b} \in \mathcal{C}_{GM}$, $i(\mathbf{a}, \mathbf{b}) = i(\mathbf{b}, \mathbf{a})$.
- (iii) For $\mathbf{a}, \mathbf{b} \in \mathcal{C}_{GM}$ and $t, s \geq 0$, $i(t\mathbf{a}, s\mathbf{b}) = ts i(\mathbf{a}, \mathbf{b})$.
- (iv) For any $y, z \in \mathcal{T}_{g,m}$,

$$i(\Psi_{x_0}(y), \Psi_{x_0}(z)) = \exp(-2\langle y | z \rangle_{x_0}),$$

where $\langle y | z \rangle_{x_0}$ is the Gromov product of y and z with basepoint x_0 with respect to the Teichmüller distance d_T , that is:

$$\langle y | z \rangle_{x_0} = \frac{1}{2}(d_T(x_0, y) + d_T(x_0, z) - d_T(y, z)).$$

In particular, we have $i(\Psi_{x_0}(y), \Psi_{x_0}(y)) = \exp(-2d_T(x_0, y))$ for $y \in \mathcal{T}_{g,m}$.

- (v) For $F, G \in \mathcal{MF} \subset \mathcal{C}_{GM}$, the value $i(F, G)$ is equal to the geometric intersection number $I(F, G)$.

Indeed, the condition (i) in Theorem 2 follows from (4.5) and (5.1). The conditions (ii) to (v) are deduced from Theorem 5.

Actually, the embedding Ψ_{x_0} in Theorem 2 is defined as

$$(1.4) \quad \Psi_{x_0}: \mathcal{T}_{g,m} \ni y \mapsto \left[\mathcal{S} \ni \alpha \mapsto \exp(-d_T(x_0, y)) \cdot \text{Ext}_y(\alpha)^{1/2} \right] \in \mathcal{C}_{GM}.$$

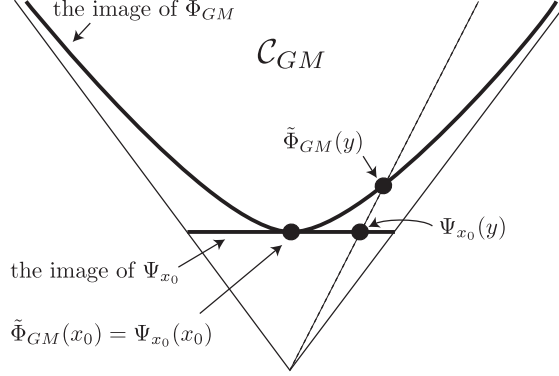
Namely,

$$(1.5) \quad \Psi_{x_0}(y) = \exp(-d_T(x_0, y)) \cdot \tilde{\Phi}_{GM}(y)$$

for all $y \in \mathcal{T}_{g,m}$. The embedding (1.4) is clearly a lift of the Gardiner-Masur embedding (1.2) of $\mathcal{T}_{g,m}$. One of advantages to use the embedding Ψ_{x_0} is that Ψ_{x_0} admits a continuous extension to $\text{cl}_{GM}(\mathcal{T}_{g,m})$, whereas $\tilde{\Phi}_{GM}$ diverges at infinity (cf. Proposition 3.1 and (4.5)).

One can easily deduce Theorem 1 from Theorem 2. Indeed, the only difference is the item (iv) in each theorem. From the homogeneity (iii) in Theorem 2, we have

$$(1.6) \quad \begin{aligned} i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(z)) &= \exp(d_T(x_0, y)) \cdot \exp(d_T(x_0, z)) \cdot i(\Psi_{x_0}(y), \Psi_{x_0}(z)) \\ &= \exp(d_T(y, z)) \end{aligned}$$

FIGURE 1. Cone \mathcal{C}_{GM} and the images of $\tilde{\Phi}_{GM}$ and Ψ_{x_0} .

for $y, z \in \mathcal{T}_{g,m}$.

1.3.4. *Hyperboloid model of Teichmüller space.* We represent the situations of our theorems schematically in Figure 1, where the image of $\tilde{\Phi}_{GM}$ looks like a hyperboloid and that of Φ_{x_0} is a section of the cone. These images touch only at the image of the basepoint. Indeed, for any $y \in \mathcal{T}_{g,m}$, $\tilde{\Phi}_{GM}(y)$ and $\Psi_{x_0}(y)$ are projectively equivalent in \mathbb{R}_+^S . From (iv) in Theorem 1, we can observe that the image under $\tilde{\Phi}_{GM}$ coincides with the “hyperboloid”

$$(1.7) \quad \{\mathfrak{a} \in \mathcal{C}_{GM} \mid i(\mathfrak{a}, \mathfrak{a}) = 1\},$$

and the boundary of the cone \mathcal{C}_{GM} is represented as the “light cone”

$$(1.8) \quad \{\mathfrak{a} \in \mathcal{C}_{GM} \mid i(\mathfrak{a}, \mathfrak{a}) = 0\}$$

from (iv) in Theorem 2 and the continuity of the intersection number on \mathcal{C}_{GM} (cf. Proposition 10.1). In the hyperboloid model above, the Teichmüller distance d_T is represented by

$$d_T(y, z) = \log i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(z))$$

from (1.6).

This observation of the “hyperboloid model” might be comparable with Bonahon’s realization of the Thurston compactification of Teichmüller space in the space of geodesic currents (cf. [5]).

1.3.5. *Extension of the Gromov product.* From Theorem 2, we conclude the following corollary, which confirms that the Gardiner-Masur boundary is a kind of a canonical boundary for the geometry of the Teichmüller distance.

Corollary 1 (Extension of the Gromov product for d_T). *For any $x_0 \in \mathcal{T}_{g,m}$, there is a unique continuous function*

$$\langle \cdot \mid \cdot \rangle_{x_0} : \text{cl}_{GM}(\mathcal{T}_{g,m}) \times \text{cl}_{GM}(\mathcal{T}_{g,m}) \rightarrow [0, +\infty]$$

such that

$$(1) \text{ for } y, z \in \mathcal{T}_{g,m},$$

$$\langle y \mid z \rangle_{x_0} = \frac{1}{2}(d_T(x_0, y) + d_T(x_0, z) - d_T(y, z)),$$

(2) For $[F], [G] \in \mathcal{PMF} \subset \partial_{GM}\mathcal{T}_{g,m}$,

$$\exp(-2\langle [F] | [G] \rangle_{x_0}) = \frac{I(F, G)}{\text{Ext}_{x_0}(F)^{1/2} \cdot \text{Ext}_{x_0}(G)^{1/2}}.$$

The conclusion in Corollary 1 is somewhat surprising, because Teichmüller space with the Teichmüller distance is believed to be a metric space with less “good natures” for geodesic triangles. For instance, it was shown that Teichmüller space is neither a metric space with Busemann negative curvature nor a Gromov hyperbolic space (cf. [26], [29] and [31]). Thus, from Corollary 1 and Liu and Su’s observation in [22], it is natural to ask the following problem.

Problem 1. *For any (proper) metric space, does the Gromov product extend continuously to the product space of two copies of the horofunction compactification?*

The affirmative answer gives a different approach to our results.

1.3.6. Rigidity theorem for mappings of bounded distortion for triangles. Our unified treatment of extremal length geometry in terms of intersection number enables us to link the geometry of the Teichmüller distance (an analytical aspect in Teichmüller theory) with the geometry on \mathcal{MF} via intersection number (a topological aspect in Teichmüller theory).

As an application of this accessibility, we will observe a rigidity theorem for certain mappings on the boundary of $\mathcal{T}_{g,m}$ at infinity, which is a kind of a generalization of Earle-Ivanov-Kra-Markovic-Royden’s characterization of the action of isometries (cf. [10], [11], [18] and [39]).

Indeed, we will deal with a *mapping of bounded distortion for triangles* which is defined as a mapping $\omega : \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ satisfying

$$\frac{1}{D_1} \langle x | y \rangle_z - D_2 \leq \langle \omega(x) | \omega(y) \rangle_{\omega(z)} \leq D_1 \langle x | y \rangle_z + D_2$$

for all $x, y, z \in \mathcal{T}_{g,m}$ and some constants $D_1, D_2 > 0$ independent of the choice of points of $\mathcal{T}_{g,m}$. A mapping $\omega' : \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ is said to be a *quasi-inverse* of a mapping $\omega : \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ if there is a constant $D_3 > 0$ such that

$$\sup_{x \in \mathcal{T}_{g,m}} \{d_T(x, \omega \circ \omega'(x)), d_T(x, \omega' \circ \omega(x))\} \leq D_3.$$

In §9.4, we prove the following.

Theorem 3 (Asymptotic Rigidity). *Suppose that the complex dimension of $\mathcal{T}_{g,m}$ is at least two. Let $\omega : \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ be a mapping of bounded distortion for triangles. Assume the following two conditions:*

- (a) *The map ω admits a continuous extension to $\partial_{GM}\mathcal{T}_{g,m}$.*
- (b) *The map ω has a quasi-inverse ω' which admits a continuous extension to $\partial_{GM}\mathcal{T}_{g,m}$.*

Then, the following hold:

- (1) *The map ω acts homeomorphically on $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}_{g,m}$ and $\omega' = \omega^{-1}$ on \mathcal{PMF} .*
- (2) *The restriction of ω to \mathcal{PMF} preserves \mathcal{S} and induces an automorphism of the complex of curves.*

From the definition, an isometry is a mapping of bounded distortion for triangles and has a quasi-inverse. Furthermore, L. Liu and W. Su showed that any isometry extends homeomorphically on $\partial_{GM}\mathcal{T}_{g,m}$ (cf. [22]). Hence, every isometry satisfies the assumptions in Theorem 3. Notice also from the definition that a mapping of bounded distortion for triangles is a quasi-isometry. However, the author does not know whether Theorem 3 holds for quasi-isometries on $\mathcal{T}_{g,m}$.

We remark that (1) in Theorem 3 holds when the complex dimension of $\mathcal{T}_{g,m}$ is equal to one. In this case, $(\mathcal{T}_{g,m}, d_T)$ is isometric to the hyperbolic plane, and both the Gardiner-Masur boundary and \mathcal{PMF} coincide with the boundary at infinity of the hyperbolic plane (cf. e.g. [32]). Hence any quasi-isometry on $(\mathcal{T}_{g,m}, d_T)$ induces a homeomorphism of \mathcal{PMF} . However, the assertion (2) does not hold because the isometry group of $(\mathcal{T}_{g,m}, d_T)$ acts transitively in this case.

1.3.7. Isometries on $\mathcal{T}_{g,m}$. In [39], H. Royden first observed a beautiful result that any biholomorphic automorphism of $\mathcal{T}_{g,m}$ is induced from an orientation preserving homeomorphism on X (see also Earle-Kra [10] and Earle-Markovic [11]). Since the Teichmüller distance coincides with the Kobayashi intrinsic distance on $\mathcal{T}_{g,m}$, Royden's result gives a characterization of certain isometries on $\mathcal{T}_{g,m}$. In [18], N. Ivanov showed that with few exception, the isometry group of $(\mathcal{T}_{g,m}, d_T)$ is isomorphic to the extended mapping class group.

Theorem 3 enable us to give an alternative approach to Earle-Ivanov-Kra-Markovic-Royden's characterization of the isometry group of $(\mathcal{T}_{g,m}, d_T)$ via the Gardiner-Masur compactification. Namely, we show the following in §9.5.

Corollary 2 (Royden [39], Earle-Kra [10], Ivanov [18], and Earle-Markovic [11]). *Suppose that $3g - 3 + m \geq 2$ and (g, m) is neither $(1, 2)$ nor $(2, 0)$. Then, the isometry group of $(\mathcal{T}_{g,m}, d_T)$ is canonically isomorphic to the extended mapping class group.*

Actually, our proof of Corollary 2 is somewhat modelled on Ivanov's proof. We outline the idea of his proof. The essential part is to show that an isometric action on $(\mathcal{T}_{g,m}, d_T)$ induces an automorphism of the complex of curves. After then, from a theorem by Ivanov, Korkmaz and Luo, we see that such an automorphism of the complex of curves is induced by an element of the extended mapping class group (cf. [17], [20] and [23]). Finally, it is checked that the action of the given isometry coincides with the action of the element of the extended mapping class group.

As noted before, our proof of Corollary 2 also follows the same line. However, our proof of the essential part above follows from Theorem 3 which holds for mappings of bounded distortion for triangles. Namely, we obtain the case of isometries as a corollary of our result. Moreover, to show the essential part above, Ivanov induces a self-homeomorphism of \mathcal{PMF} as Theorem 3. To do this, he identifies \mathcal{PMF} with the unit sphere in the tangent space, and defines the self-homeomorphism by passing the "exponential maps" (cf. the discussion after the proof of Lemma 5.2 in [18]). Therefore, it seems to be essential in his proof that the mapping which we treat is an isometry.

Comments on the exceptional cases. Suppose first that $(g, m) = (1, 2)$. It is known that the canonical homomorphism from the extended mapping class group on $X_{1,2}$ to the isometry group is neither injective nor surjective. Indeed, by Proposition 1.3 in [10], $\mathcal{T}_{1,2}$ admits a biholomorphic mapping to the Teichmüller space $\mathcal{T}_{0,5}$ of a sphere $X_{0,5}$ with five punctures which is induced by the quotient mapping

$X_{1,2} \rightarrow X_{0,5}$ of the action of the hyperelliptic involution (double branched points are considered as punctures). Hence, from Corollary 2, the isometry group of $\mathcal{T}_{1,2}$ is isometric to the extended mapping class group $\text{Mod}^*(X_{0,5})$ of $X_{0,5}$ since the Teichmüller distance coincides with the Kobayashi distance. Therefore, the canonical homomorphism from the extended mapping class group $\text{Mod}^*(X_{1,2})$ to the isometry group of $\mathcal{T}_{1,2}$ is not surjective (cf. Corollary 3 in §4.3 of [10]). By a theorem due (independently) to Birman and Viro, the hyperelliptic involution of $X_{1,2}$ fixes every non-trivial and non-peripheral simple closed curves on $X_{1,2}$ (cf. [3]). Hence, the hyperelliptic involution acts trivially on $\mathcal{T}_{1,2}$ and the canonical homomorphism is not injective (cf. [10]).

When $(g, m) = (2, 0)$, any automorphism of the complex of curves induces a homeomorphism on $X_{2,0}$. However, the hyperelliptic involution fixes every non-trivial simple closed curves on $X_{2,0}$ and hence the action of the extended mapping class group is not faithful (cf. e.g. §9.5.2). In fact, it is known that the hyperelliptic involution generates the kernel of the canonical homomorphism.

Comments on the characterization of biholomorphisms. The problem to characterizing isometries and biholomorphisms are formulated for Teichmüller spaces of arbitrary Riemann surfaces. In the case where the Teichmüller space is of infinite dimensional, Earle and Gardiner [9] gave the adjointness theorem which asserts that any \mathbb{C} -isometry between the tangent spaces of Teichmüller spaces induces a \mathbb{C} -isometry of spaces of holomorphic quadratic differentials of corresponding Riemann surfaces. Earle and Gardiner also obtained that the characterization for Riemann surfaces of topologically finite type. In [21], N. Lakic obtained the characterization for Riemann surfaces of finite genus. Finally, in [24], Markovic settled the characterization for biholomorphisms of Teichmüller space of arbitrary Riemann surfaces.

1.4. Plan of this paper. This paper is organized as follows. In §§2 and 3, we recall basic notions in Teichmüller theory and known results for the Gardiner-Masur compactification. Especially, we will recall a continuous function \mathcal{E}_p on \mathcal{MF} associated to $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$ (cf. [33]). Notice that the function \mathcal{E}_p is dependent on the choice of the basepoint $x_0 \in \mathcal{T}_{g,m}$. Namely, $\mathcal{E}_p = \mathcal{E}_p^{x_0}$ (cf. §3.1).

In §4, we define the cones which are essential objects in this paper. We also define the (topological) *models* of cones, and canonical identifications between cones and their models. By definition, the model of the cone canonically contains $\text{cl}_{GM}(\mathcal{T}_{g,m})$ (cf. §4.2). We use such models when we develop an argument which depends on the choice of the basepoint of $\mathcal{T}_{g,m}$.

From §5 to §8, we devote to define the intersection number on the cone \mathcal{C}_{GM} . The strategy to define the intersection number on \mathcal{C}_{GM} is simple : We first recognize the value $\mathcal{E}_p(F)$ as the intersection number between $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$ and $F \in \mathcal{MF}$, and then, we extend it to the product $\mathcal{C}_{GM} \times \mathcal{C}_{GM}$.

In §5, we define the extremal length $\text{Ext}^{x_0}(\cdot)$ and the intersection number $i_{x_0}(\cdot, \cdot)$ associated to the basepoint x_0 on a part of the models. The definition of this “new” extremal length is motivated by Minsky’s inequality (cf. (2.6)). Namely, from Minsky’s inequality, we have the following formula

$$(1.9) \quad \text{Ext}_y(G) = \sup_{F \in \mathcal{MF} - \{0\}} \frac{I(G, F)^2}{\text{Ext}_y(F)}$$

for $G \in \mathcal{MF}$. Unfortunately, the equation (1.9) can not be adopted as the definition of the extremal length for measured foliations because the right-hand side is calculated by using the extremal length. However, for extending the extremal length on our cone \mathcal{C}_{GM} , we will adopt the formula (1.9). Thus, we first define the intersection number between elements of \mathcal{C}_{GM} and measured foliations (§5.1), and after then, we define the extremal length for elements of \mathcal{C}_{GM} on marked Riemann surfaces (§5.2).

The definitions of our extremal length function and intersection number function given in the previous paragraph depend on the choice of the basepoint. Hence, our discussion will be given on the models defined above. It might look to be awkward because our main result, Theorem 1, is independent of the choice of basepoint. However, because our function \mathcal{E}_p , which is a basic object in the starting point, depends on the choice of the basepoint, we are compelled to begin with a basepoint-depending argument. Actually, later on, our intersection number and extremal length will be shown to be *intrinsic* in the sense that each of them is represented as the composition of a function on the cone and the identification between the cone and the model defined in §4 (cf. §5.3 and §8.2).

In §6, we discuss the topology of models of cones. The cone is locally compact and metrizable. Furthermore, we see that the extremal length function is a proper function (cf. Proposition 6.1). We also give a system of neighborhoods which will be used for showing the equicontinuity of the family of continuous functions discussed in §7 (cf. §6.2).

In §§7 and 8, We will obtain the intersection number on the cone by extending the functions defined in the previous sections. In the beginning of §7, we define $\mathcal{E}_\eta(\zeta)$ where ζ and η are in a part of the model of the cone. The function $\mathcal{E}_\eta(\zeta)$ is an extended notion of the function \mathcal{E}_p which we noted before (cf. (7.1)). In §7.1, we notice a relation between our function $\mathcal{E}_\eta(\zeta)$ and the Gromov product for d_T (cf. (7.3)). As discussed above, we will show that the family $\{\mathcal{E}_\eta\}_\eta$ is a normal family of continuous functions on a part of the cone in §7.2. Indeed, our intersection number is defined as the limit of such a normal family in §8.1. The proof of the extension is completed in §8.2 by checking that the intersection number is intrinsic. We will prove Corollary 1 in §8.3.

In the final section §9, we prove Theorem 3 and give an alternative approach to Earle-Kra-Ivanov-Markovic-Royden's characterization in Corollary 2. The strategy to prove Theorem 3 is to investigate the *null space* $\mathcal{N}(\mathfrak{a})$ for $\mathfrak{a} \in \mathcal{C}_{GM}$ (cf. §9.2).

From Corollary 1, when a mapping ω of bounded distortion for triangles admits a continuous extension to $\partial_{GM}\mathcal{T}_{g,m}$, we observe that for $p, q \in \partial_{GM}\mathcal{T}_{g,m}$, the intersection number between p and q is zero if and only if so is that between $\omega(p)$ and $\omega(q)$ (cf. Proposition 9.2). We will also characterize uniquely ergodic measured foliations (in \mathcal{C}_{GM}) in terms of their null spaces (cf. §9.3.1). Using the observation and the characterization, we deduce that ω induces a self-homeomorphism of \mathcal{PMF} , which implies (1) of Theorem 3 (cf. §9.3). In the proof of (2) of Theorem 3, the author is impressed with that in Theorem A in Ivanov [18].

In the last section, we will discuss the hyperboloid model on Teichmüller space with the Teichmüller distance.

2. TEICHMÜLLER THEORY

2.1. Teichmüller space. The *Teichmüller space* Throughout this paper, we fix a Riemann surface X of analytically finite type (g, m) with $2g - 2 + m > 0$. The *Teichmüller space* $\mathcal{T}_{g,m}$ of Riemann surfaces of analytically finite type (g, m) is the set of equivalence classes of marked Riemann surfaces (Y, f) where Y is a Riemann surface and $f : X \rightarrow Y$ a quasiconformal mapping. Two marked Riemann surfaces (Y_1, f_1) and (Y_2, f_2) are said to be *Teichmüller equivalent* if there is a conformal mapping $h : Y_1 \rightarrow Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$.

Teichmüller space $\mathcal{T}_{g,m}$ has a canonical complete distance, called the *Teichmüller distance* d_T , which is defined by

$$(2.1) \quad d_T(y_1, y_2) = \frac{1}{2} \log \inf \{K(h) \mid h \text{ is q.c. homotopic to } f_2 \circ f_1^{-1}\}$$

for $y_i = (Y_i, f_i) \in \mathcal{T}_{g,m}$ ($i = 1, 2$) and $K(h)$ the maximal dilatation of h . The Teichmüller distance has the geometric representation called *Kerckhoff's formula* which we recall in §2.4.

Convention 1. As we noted in Introduction, in some part of this paper, we will give a basepoint-depending argument. For instance, our function \mathcal{E}_p defined in §3 is determined up to multiplications by positive constants, which depends on the choice of the basepoint. Unless otherwise noted, throughout this paper, we consider $x_0 = (X, id) \in \mathcal{T}_{g,m}$ as a basepoint.

2.2. Measured foliations. Denote by $\mathbb{R}_+ \otimes \mathcal{S}$ the set of formal products $t\alpha$ where $t \geq 0$ and $\alpha \in \mathcal{S}$. The set $\mathbb{R}_+ \otimes \mathcal{S}$ is embedded into $\mathbb{R}_+^{\mathcal{S}}$ by

$$(2.2) \quad \mathbb{R}_+ \otimes \mathcal{S} \ni t\alpha \mapsto [\mathcal{S} \ni \beta \mapsto tI(\alpha, \beta)] \in \mathbb{R}_+^{\mathcal{S}},$$

where $I(\cdot, \cdot)$ is the *geometric intersection number*. We topologize $\mathbb{R}_+^{\mathcal{S}}$ with the pointwise convergence. The *space \mathcal{MF} of measured foliations on X* is the closure of the image of the mapping (2.2). The intersection number of any two weighted curves in $\mathbb{R}_+ \otimes \mathcal{S}$ is defined by $I(t\alpha, s\beta) = tsI(\alpha, \beta)$. It is known that the intersection number function extends continuously on $\mathcal{MF} \times \mathcal{MF}$ (cf. [37]).

The positive numbers $\mathbb{R}_{>0}$ acts on $\mathbb{R}_+^{\mathcal{S}}$ by multiplication. Let

$$(2.3) \quad \text{proj} : \mathbb{R}_+^{\mathcal{S}} - \{0\} \rightarrow \mathbb{P}\mathbb{R}_+^{\mathcal{S}} = (\mathbb{R}_+^{\mathcal{S}} - \{0\})/\mathbb{R}_{>0}$$

be the quotient mapping. The space \mathcal{PMF} of *projective measured foliations* is defined to be the quotient

$$\mathcal{PMF} = \text{proj}(\mathcal{MF} - \{0\}) = (\mathcal{MF} - \{0\})/\mathbb{R}_{>0}.$$

It is known that \mathcal{MF} and \mathcal{PMF} are homeomorphic to $\mathbb{R}^{6g-6+2n}$ and $S^{6g-7+2n}$ respectively (cf. [7]).

2.3. Extremal length. For $y = (Y, f) \in \mathcal{T}_{g,m}$ and $\alpha \in \mathcal{S}$, the *extremal length of α on y* is defined by

$$(2.4) \quad \text{Ext}_y(\alpha) = 1/\sup_A \{\text{Mod}(A) \mid A \subset Y \text{ and the core is homotopic to } f(\alpha)\},$$

where $\text{Mod}(A)$ is the *modulus* of an annulus A , which is equal to $(\log r)/2\pi$ if A is conformally equivalent to a round annulus $\{1 < |z| < r\}$.

For $t\alpha \in \mathbb{R}_+ \otimes \mathcal{S}$, we set

$$\text{Ext}_y(t\alpha) = t^2 \text{Ext}_y(\alpha).$$

In [19], Kerckhoff showed that the extremal length function extends continuously on \mathcal{MF} . Let

$$(2.5) \quad \mathcal{MF}_1 = \{F \in \mathcal{MF} \mid \text{Ext}_{x_0}(F) = 1\}.$$

The extremal length of measured foliations satisfies the following inequality, which is called *Minsky's inequality*:

$$(2.6) \quad I(F, G)^2 \leq \text{Ext}_y(F) \cdot \text{Ext}_y(G)$$

for all $y \in \mathcal{T}_{g,m}$ and $F, G \in \mathcal{MF}$ (cf. [36]). Minsky's inequality is sharp in the sense that for any $y \in \mathcal{T}_{g,m}$ and $F \in \mathcal{MF} - \{0\}$, there is a unique $G \in \mathcal{MF} - \{0\}$ up to positive multiple such that

$$(2.7) \quad I(F, G)^2 = \text{Ext}_y(F) \cdot \text{Ext}_y(G).$$

Furthermore, such a pair F and G of measured foliations are realized by the horizontal and vertical foliations of a holomorphic quadratic differential on a marked Riemann surface y , and vice versa (cf. [13]).

2.4. Kerckhoff's formula. In [19], Kerckhoff also observed that the Teichmüller distance is represented by the following formula:

$$(2.8) \quad d_T(y, z) = \frac{1}{2} \log \sup_{F \in \mathcal{MF} - \{0\}} \frac{\text{Ext}_y(F)}{\text{Ext}_z(F)} = \frac{1}{2} \log \max_{F \in \mathcal{MF}_1} \frac{\text{Ext}_y(F)}{\text{Ext}_z(F)}.$$

In fact, for any $y_1, y_2 \in \mathcal{T}_{g,m}$, there is a unique pair (F, G) of measured foliations in \mathcal{MF}_1 such that

$$(2.9) \quad \frac{\text{Ext}_{y_1}(F)}{\text{Ext}_{y_2}(F)} = \frac{\text{Ext}_{y_2}(G)}{\text{Ext}_{y_1}(G)} = e^{2d_T(y_1, y_2)}.$$

3. THE GARDINER-MASUR CLOSURE

In this section, we recall basic properties of the Gardiner-Masur closure. In §3.1 below, we assign a continuous function \mathcal{E}_p on \mathcal{MF} to all point p in the Gardiner-Masur closure. As noted in Convention 1, the function \mathcal{E}_p depends on the choice of basepoint x_0 . When we emphasis the dependence, we write $\mathcal{E}_p^{x_0}$ instead of \mathcal{E}_p . As we will observe in Proposition 8.1 later, the value $\mathcal{E}_p(F)$ is recognized as the intersection number between p and $F \in \mathcal{MF}$ associated to x_0 (see also §5.1).

3.1. Function \mathcal{E}_p . For $y \in \mathcal{T}_{g,m}$, we define a continuous function $\mathcal{E}_y = \mathcal{E}_y^{x_0}$ on \mathcal{MF} by

$$(3.1) \quad \mathcal{E}_y(F) = \mathcal{E}_y^{x_0}(F) = \left\{ \frac{\text{Ext}_y(F)}{K_y} \right\}^{1/2}$$

where $K_y = \exp(2d_T(x_0, y))$. In [33], the author showed that for any $p \in \partial_{GM} \mathcal{T}_{g,m}$, there is a continuous function \mathcal{E}_p on \mathcal{MF} such that

- (E1) the projective class of the assignment $\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)$ is equal to p ;
- (E2) if a sequence $\{y_n\}_{n=1}^\infty$ converges to $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$, there are $t_0 > 0$ and a subsequence $\{y_{n_j}\}_j$ such that $\mathcal{E}_{y_{n_j}}$ converges to $t_0 \mathcal{E}_p$ uniformly on any compact set of \mathcal{MF} .

Notice that from Observations (E1) and (E2), the function \mathcal{E}_p is *projectively* determined. Namely, for all $t > 0$ and $p \in \partial_{GM} \mathcal{T}_{g,m}$, the product $t \mathcal{E}_p$ also satisfies (E1) and (E2) above. We first sharpen the condition (E2) above as follows (cf. [35])

Proposition 3.1. *For any $p \in \partial_{GM}\mathcal{T}_{g,m}$, one can choose \mathcal{E}_p appropriately such that a function*

$$\text{cl}_{GM}(\mathcal{T}_{g,m}) \times \mathcal{MF} \ni (p, F) \mapsto \mathcal{E}_p(F)$$

is continuous.

Proof. We normalize \mathcal{E}_p such that

$$(3.2) \quad \max_{F \in \mathcal{MF}_1} \mathcal{E}_p(F) = 1$$

(cf. (2.5)). Notice from (2.9) that $\max_{F \in \mathcal{MF}_1} \mathcal{E}_y(F) = 1$ for all $y \in \mathcal{T}_{g,m}$. Let $\{y_n\}_{n=1}^\infty$ be a sequence that converges to $p \in \partial_{GM}\mathcal{T}_{g,m}$. From the condition (E2) above, there are a subsequence $\{y_{n_j}\}_j$ and $t_0 > 0$ such that $\mathcal{E}_{y_{n_j}}$ converges to $t_0\mathcal{E}_p$ uniformly on any compact set of \mathcal{MF} , and hence

$$1 = \max_{F \in \mathcal{MF}_1} \mathcal{E}_{y_{n_j}}(F) \rightarrow t_0 \max_{F \in \mathcal{MF}_1} \mathcal{E}_p(F) = t_0.$$

This implies that \mathcal{E}_{y_n} converges to \mathcal{E}_p on any compact set of \mathcal{MF} . \square \square

Convention 2. *In what follows, we normalize \mathcal{E}_p as in (3.2) for all $p \in \partial_{GM}\mathcal{T}_{g,m}$.*

3.2. Properties of \mathcal{E}_p . Here we give some properties of \mathcal{E}_p .

Lemma 3.1. *For $G \in \mathcal{MF}$, we have*

$$\mathcal{E}_{[G]}(F) = \frac{I(F, G)}{\text{Ext}_{x_0}(G)^{1/2}}$$

for all $F \in \mathcal{MF}$.

Proof. By definition, there is a positive number t_0 such that $\mathcal{E}_{[G]}(F) = t_0 I(F, G)$ for all $F \in \mathcal{MF}$. By (2.7), we obtain

$$1 = \max_{F \in \mathcal{MF}_1} \mathcal{E}_{[G]}(F) = t_0 \max_{F \in \mathcal{MF}_1} I(F, G) = t_0 \text{Ext}_{x_0}(G)^{1/2}$$

(cf. Convention 2), and we are done. \square \square

The following is proven in [35]. However, for completeness, we shall give a proof.

Proposition 3.2. *For $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$, the following are equivalent.*

- (1) $p \in \partial_{GM}\mathcal{T}_{g,m}$;
- (2) *there is an $F \in \mathcal{MF} - \{0\}$ with $\mathcal{E}_p(F) = 0$.*

Proof. By applying Minsky's inequality, one can easily see that for $y \in \mathcal{T}_{g,m}(X)$, $\mathcal{E}_y(F) = 0$ if and only if $F = 0$. Therefore, (2) implies (1).

Conversely, let $\{y_n\}_{n=1}^\infty$ be a sequence converging to $p \in \partial_{GM}\mathcal{T}_{g,m}$. Then, by a theorem of Bers [2], there is a curve $\alpha_n \in \mathcal{S}$ such that $\text{Ext}_{y_n}(\alpha_n) = O(1)$ as $n \rightarrow \infty$. By taking a subsequence if necessary, we find $t_n > 0$ such that $t_n \alpha_n \rightarrow F \in \mathcal{MF} - \{0\}$. This means that $t_n = O(1)$ since $\text{Ext}_{x_0}(t_n \alpha_n) \rightarrow \text{Ext}_{x_0}(F)$ and $\text{Ext}_{x_0}(\alpha_n)$ is bounded below by a uniform positive constant. From the property (E2) above, we have

$$\mathcal{E}_p(F) = \lim_{n \rightarrow \infty} \frac{t_n \text{Ext}_{y_n}(\alpha_n)^{1/2}}{K_{y_n}^{1/2}} = 0$$

as $n \rightarrow \infty$ and we are done. \square \square

4. CONES \mathcal{C}_{GM} , \mathcal{T}_{GM} AND $\tilde{\partial}_{GM}$

4.1. **Cones.** Define

$$(4.1) \quad \mathcal{C}_{GM} = \text{proj}^{-1}(\text{cl}_{GM}(\mathcal{T}_{g,m})) \cup \{0\} \subset \mathbb{R}_+^S$$

$$(4.2) \quad \mathcal{T}_{GM} = \text{proj}^{-1}(\mathcal{T}_{g,m}) \cup \{0\} \subset \mathbb{R}_+^S$$

$$(4.3) \quad \tilde{\partial}_{GM} = \text{proj}^{-1}(\partial_{GM}\mathcal{T}_{g,m}) \cup \{0\} \subset \mathcal{C}_{GM} \subset \mathbb{R}_+^S.$$

We topologize \mathcal{C}_{GM} , \mathcal{T}_{GM} and $\tilde{\partial}_{GM}$ with the topology induced from \mathbb{R}_+^S . Notice that \mathcal{MF} is contained in $\tilde{\partial}_{GM}$ as a closed subset since $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}_{g,m}$. In particular, from the definition, any $G \in \mathcal{MF}$ is nothing other than an assignment

$$(4.4) \quad \mathcal{S} \ni \alpha \mapsto I(\alpha, G).$$

We define an embedding $\Psi_{x_0} : \text{cl}_{GM}(\mathcal{T}_{g,m}) \rightarrow \mathcal{C}_{GM}$ by

$$(4.5) \quad \Psi_{x_0} : \text{cl}_{GM}(\mathcal{T}_{g,m}) \ni p \mapsto [\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)] \in \mathcal{C}_{GM}.$$

By definition, the embedding (4.5) is a lift of the embedding (1.2). However, it depends on the basepoint x_0 .

4.2. **Models of \mathcal{C}_{GM} , \mathcal{T}_{GM} and $\tilde{\partial}_{GM}$.** We define models of cones by

$$\mathbf{MC}_{GM} = \text{cl}_{GM}(\mathcal{T}_{g,m}) \times \mathbb{R}_+ / (\text{cl}_{GM}(\mathcal{T}_{g,m}) \times \{0\})$$

$$\mathbf{MT}_{GM} = \mathcal{T}_{g,m} \times \mathbb{R}_+ / (\mathcal{T}_{g,m} \times \{0\})$$

$$\mathbf{M}\tilde{\partial}_{GM} = \partial_{GM}\mathcal{T}_{g,m} \times \mathbb{R}_+ / (\partial_{GM}\mathcal{T}_{g,m} \times \{0\})$$

$$\mathbf{MF} = \mathcal{PMF} \times \mathbb{R}_+ / (\mathcal{PMF} \times \{0\})$$

$$\mathbf{MF}_1 = \mathcal{PMF} \times \{1\}.$$

By definition, \mathbf{MC}_{GM} is a union of \mathbf{MT}_{GM} and $\mathbf{M}\tilde{\partial}_{GM}$. Furthermore, $\mathbf{MF}_1 \subset \mathbf{MF} \subset \mathbf{M}\tilde{\partial}_{GM}$ since $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}_{g,m}$. In this setting, we often identify $\text{cl}_{GM}(\mathcal{T}_{g,m})$ with the slice $\text{cl}_{GM}(\mathcal{T}_{g,m}) \times \{1\}$ of \mathbf{MC}_{GM} .

We abbreviate the point $(p, t) \in \mathbf{MC}_{GM}$ to tp . We denote $1p$ by p for the simplicity. For $s \geq 0$ and $\zeta = tp \in \mathbf{MC}_{GM}$ with $t \geq 0$ and $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$, we define the multiplication $s\zeta$ by

$$s\zeta = (st)p.$$

From Proposition 3.1, the embedding (4.5) is continuous. Therefore, we have a continuous bijection

$$\tilde{\Psi}_{x_0} : \mathbf{MC}_{GM} \rightarrow \mathcal{C}_{GM}$$

by

$$(4.6) \quad \tilde{\Psi}_{x_0}(tp) = \tilde{\Psi}_{x_0}(p, t) = t \cdot \Psi_{x_0}(p) = [\mathcal{S} \ni \alpha \mapsto t\mathcal{E}_p(\alpha)].$$

By definition, $\tilde{\Psi}_{x_0}$ is homogeneous:

$$\tilde{\Psi}_{x_0}(t\zeta) = t\tilde{\Psi}_{x_0}(\zeta)$$

for $t \geq 0$ and $\zeta \in \mathbf{MC}_{GM}$. Since $\text{cl}_{GM}(\mathcal{T}_{g,m})$ is compact, \mathbb{R}_+ is locally compact and \mathcal{C}_{GM} is Hausdorff, the bijection $\tilde{\Psi}_{x_0}$ is a homeomorphism. It follows from Lemma 3.1 that

$$(4.7) \quad \tilde{\Psi}_{x_0}(s[F]) = s \text{Ext}_{x_0}(F)^{-1/2} \cdot F \in \mathcal{MF}$$

for $s[F] \in \mathbf{MF}$ and hence $\tilde{\Psi}_{x_0}(\mathbf{MF}) = \mathcal{MF}$.

Lemma 4.1 (Image of \mathbf{MF}_1). *For $[G] \in \mathbf{MF}_1$, we have $\Psi_{x_0}([G]) \in \mathcal{MF}_1 \subset \tilde{\partial}_{GM}$.*

Proof. From Lemma 3.1, the projective class $[G] \in \mathbf{MF}_1$ corresponds to the assignment

$$(4.8) \quad \mathcal{S} \ni \alpha \mapsto \frac{I(\alpha, G)}{\text{Ext}_{x_0}(G)^{1/2}} = I\left(\alpha, \text{Ext}_{x_0}(G)^{-1/2} \cdot G\right).$$

Therefore, we deduce from (4.4) that $\Psi_{x_0}([G]) \in \mathcal{MF}_1$. \square \square

Remark 4.1. *From the identification (4.6), we can recognize \mathcal{C}_{GM} , \mathcal{T}_{GM} and $\tilde{\partial}_{GM}$ as cones with slices $\text{cl}_{GM}(\mathcal{T}_{g,m})$, $\mathcal{T}_{g,m}$ and $\partial_{GM}\mathcal{T}_{g,m}$, respectively. Notice that this identification depends on the choice of the basepoint x_0 , because so is the embedding Ψ_{x_0} in (4.5).*

5. INTERSECTION NUMBER AND EXTREMAL LENGTH ASSOCIATED TO A POINT

In this section, we define the intersection number on $\mathbf{MC}_{GM} \times \mathbf{MF}$ and the extremal length for elements in \mathbf{MC}_{GM} associated to the basepoint x_0 . We will extend the intersection number given here to the whole $\mathbf{MC}_{GM} \times \mathbf{MC}_{GM}$ in §8.1.

5.1. Intersection number associated to the basepoint. For $\zeta = tp \in \mathbf{MC}_{GM}$ ($t \geq 0$ and $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$) and $\eta \in \mathbf{MF}$, we define the *intersection number associated to the basepoint x_0* by

$$(5.1) \quad i_{x_0}(\zeta, \eta) = i_{x_0}(tp, \eta) = t \mathcal{E}_p(\tilde{\Psi}_{x_0}(\eta)) = t \mathcal{E}_p^{x_0}(\tilde{\Psi}_{x_0}(\eta)).$$

The intersection number (5.1) depends on the basepoint x_0 . Indeed, By (4.7), we have

$$(5.2) \quad \begin{aligned} i_{x_0}(ty, s[F]) &= t \mathcal{E}_y(\tilde{\Psi}_{x_0}(s[F])) \\ &= t \left\{ \frac{\text{Ext}_y(s \text{Ext}_{x_0}(F)^{-1/2} \cdot F)}{K_y} \right\}^{1/2} \\ &= ts \cdot e^{-d_T(x_0, y)} \left(\frac{\text{Ext}_y(F)}{\text{Ext}_{x_0}(F)} \right)^{1/2} \end{aligned}$$

for $ty \in \mathbf{MT}_{GM}$ and $s[F] \in \mathbf{MF}$.

By (4.6), $\zeta \in \mathbf{MC}_{GM}$ corresponds to the function

$$(5.3) \quad \mathcal{S} \ni \alpha \mapsto i_{x_0}(\zeta, \tilde{\Psi}_{x_0}^{-1}(\alpha))$$

in \mathcal{C}_{GM} via $\tilde{\Psi}_{x_0}$. From Proposition 3.1, the assignment

$$\mathbf{MC}_{GM} \times \mathbf{MF} \ni (\zeta, \eta) \mapsto i_{x_0}(\zeta, \eta)$$

is continuous. Furthermore, the intersection number (5.1) is *homogeneous* since

$$\begin{aligned} i_{x_0}(s_1\zeta, s_2\eta) &= i_{x_0}((s_1t)p, s_2\eta) = (s_1t) \mathcal{E}_p(\tilde{\Psi}_{x_0}(s_2\eta)) \\ &= s_1s_2 \cdot t \mathcal{E}_p(\tilde{\Psi}_{x_0}(\eta)) = s_1s_2 i_{x_0}(\zeta, \eta) \end{aligned}$$

where $s_1, s_2 \geq 0$, $\zeta = tp$ with $t \geq 0$ and $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$, and $\eta \in \mathbf{MF}$.

Proposition 5.1 (Intersection number on \mathcal{MF}). *The intersection number function (5.1) coincides with the original intersection number function on $\mathcal{MF} \times \mathcal{MF}$ via $\tilde{\Psi}_{x_0}$. Namely, when $G = \tilde{\Psi}_{x_0}(\zeta)$ and $F = \tilde{\Psi}_{x_0}(\eta)$ with $\zeta, \eta \in \mathbf{MF}$,*

$$i_{x_0}(\zeta, \eta) = I(G, F).$$

Proof. Notice from (4.7) that $\zeta = \tilde{\Psi}_{x_0}^{-1}(G) = \text{Ext}_{x_0}(G)^{1/2} \cdot [G]$. By Lemma 3.1, we have

$$\begin{aligned} i_{x_0}(\zeta, \eta) &= i_{x_0}(\text{Ext}_{x_0}(G)^{1/2} \cdot [G], \eta) \\ &= \text{Ext}_{x_0}(G)^{1/2} \mathcal{E}_{[G]}(\tilde{\Psi}_{x_0}(\eta)) \\ &= \text{Ext}_{x_0}(G)^{1/2} \mathcal{E}_{[G]}(F) \\ &= I(G, F), \end{aligned}$$

which implies what we wanted. \square \square

5.2. Extremal length on MC_{GM} associated to the basepoint. For $\zeta \in \text{MC}_{GM}$, we define the *extremal length of ζ on $ty \in \text{MT}_{GM}$ associated to the basepoint x_0* by

$$(5.4) \quad \mathcal{E}xt_{ty}^{x_0}(\zeta) = t^2 \cdot \max_{\eta \in \text{MF}_1} \frac{i_{x_0}(\zeta, \eta)^2}{\text{Ext}_y(\tilde{\Psi}_{x_0}(\eta))} = t^2 \cdot \sup_{F \in \text{MF} - \{0\}} \frac{i_{x_0}(\zeta, \eta)^2}{\text{Ext}_y(\tilde{\Psi}_{x_0}(\eta))}.$$

As noted in §1.4, this definition is given by the imitating the formula (1.9). By definition, $\mathcal{E}xt_{ty}^{x_0}(\cdot)$ is homogeneous and satisfies

$$(5.5) \quad i_{x_0}(\zeta, \eta)^2 \leq \mathcal{E}xt_y^{x_0}(\zeta) \cdot \text{Ext}_y(\Psi_{x_0}(\eta))$$

for all $y \in \mathcal{T}_{g,m}$, $\zeta \in \text{MC}_{GM}$ and $\eta \in \text{MF}$. Since MF_1 is compact, for every $\zeta \in \text{MC}_{GM}$, there is an $\eta \in \text{MF} - \{0\}$ such that

$$\mathcal{E}xt_{ty}^{x_0}(\zeta) = t^2 \frac{i_{x_0}(\zeta, \eta)^2}{\text{Ext}_y(\Psi_{x_0}(\eta))}$$

or

$$(5.6) \quad i_{x_0}(\zeta, \eta)^2 = \mathcal{E}xt_{ty}^{x_0}(\zeta) \cdot \text{Ext}_y(\Psi_{x_0}(\eta)).$$

5.2.1. Basic properties. We can easily see the following.

Lemma 5.1. *For $y \in \mathcal{T}_{g,m}$, the following two properties hold.*

(1) *For $ty, sz \in \text{MT}_{GM}$ with $t, s \geq 0$ and $y, z \in \mathcal{T}_{g,m}$,*

$$\mathcal{E}xt_{ty}^{x_0}(sz) = t^2 s^2 \exp(-2d_T(x_0, z) + 2d_T(y, z)).$$

(2) *For $\zeta \in \text{MF}$ and $y \in \mathcal{T}_{g,m}$,*

$$\mathcal{E}xt_y^{x_0}(\zeta) = \text{Ext}_y(\tilde{\Psi}_{x_0}(\zeta)).$$

Proof. (1) Since $K_z = \exp(2d_T(x_0, z))$, from Kerckhoff's formula, we have

$$\begin{aligned} \mathcal{E}xt_{ty}^{x_0}(sz) &= t^2 \cdot \sup_{\eta \in \text{MF} - \{0\}} \frac{i_{x_0}(sz, \eta)^2}{\text{Ext}_y(\Psi_{x_0}(\eta))} = t^2 s^2 \sup_{F \in \text{MF} - \{0\}} \frac{\text{Ext}_z(F)}{K_z \text{Ext}_y(F)} \\ &= t^2 s^2 \exp(-2d_T(x_0, z) + 2d_T(y, z)). \end{aligned}$$

(2) This follows from Proposition 5.1 and the sharpness of Minsky's inequality. \square \square

We notice the following non-triviality of extremal length (5.4).

Lemma 5.2 (Non-triviality). *Let $\zeta \in \text{MC}_{GM}$. If $\mathcal{E}xt_y^{x_0}(\zeta) = 0$ for some $y \in \mathcal{T}_{g,m}$, then $\zeta = 0$.*

Proof. Take $t \geq 0$ and $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$ with $\zeta = tp$. Suppose $\mathcal{E}xt_y^{x_0}(\zeta) = 0$. From (5.1), we have

$$\begin{aligned} 0 = \mathcal{E}xt_y^{x_0}(\zeta) &= \sup_{\eta \in \text{MF} - \{0\}} \frac{i_{x_0}(\zeta, \eta)^2}{\text{Ext}_y(\tilde{\Psi}_{x_0}(\eta))} = \sup_{\eta \in \text{MF} - \{0\}} \frac{\mathcal{E}_\zeta(\tilde{\Psi}_{x_0}(\eta))^2}{\text{Ext}_y(\tilde{\Psi}_{x_0}(\eta))} \\ &= \sup_{F \in \mathcal{MF} - \{0\}} \frac{\mathcal{E}_\zeta(F)^2}{\text{Ext}_y(F)} = \sup_{F \in \mathcal{MF} - \{0\}} \frac{t^2 \mathcal{E}_p(F)^2}{\text{Ext}_y(F)}. \end{aligned}$$

Therefore, we obtain

$$t \mathcal{E}_p(F) = 0$$

for all $F \in \mathcal{MF} - \{0\}$. On the other hand, since $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$, $\mathcal{E}_p(\alpha) \neq 0$ for some $\alpha \in \mathcal{S}$, and we get $t = 0$. Therefore, $\zeta = tp = 0$. \square \square

5.2.2. Continuity. Notice that the extremal length given in (5.4) satisfies the *distortion property*:

$$(5.7) \quad e^{-2d_T(y_1, y_2)} \mathcal{E}xt_{y_1}^{x_0}(\zeta) \leq \mathcal{E}xt_{y_2}^{x_0}(\zeta) \leq e^{2d_T(y_1, y_2)} \mathcal{E}xt_{y_1}^{x_0}(\zeta)$$

for $y_1, y_2 \in \mathcal{T}_{g,m}$ and $\zeta \in \text{MC}_{GM}$. Indeed, since $\tilde{\Psi}_{x_0}(\eta) \in \mathcal{MF}$ for $\eta \in \text{MF}$, we have

$$\text{Ext}_{y_1}(\tilde{\Psi}_{x_0}(\eta)) \geq e^{-2d_T(y_1, y_2)} \text{Ext}_{y_2}(\tilde{\Psi}_{x_0}(\eta))$$

for all $\eta \in \text{MF}$. Therefore, we obtain

$$\begin{aligned} \mathcal{E}xt_{y_2}^{x_0}(\zeta) &= \sup_{\eta \in \text{MF} - \{0\}} \frac{i_{x_0}(\zeta, \eta)^2}{\text{Ext}_{y_2}(\tilde{\Psi}_{x_0}(\eta))} \\ &\leq e^{2d_T(y_1, y_2)} \sup_{\eta \in \text{MF} - \{0\}} \frac{i_{x_0}(\zeta, \eta)^2}{\text{Ext}_{y_1}(\tilde{\Psi}_{x_0}(\eta))} \\ &= e^{2d_T(y_1, y_2)} \mathcal{E}xt_{y_1}^{x_0}(\zeta). \end{aligned}$$

Lemma 5.3 (Continuity). *The function*

$$(5.8) \quad \text{M}\mathcal{T}_{GM} \times \text{MC}_{GM} \ni (ty, \zeta) \mapsto \mathcal{E}xt_{ty}^{x_0}(\zeta)$$

is continuous.

Proof. When a sequence $\{\zeta_n\}_{n=1}^\infty$ in MC_{GM} converges to ζ , \mathcal{E}_{ζ_n} converges to \mathcal{E}_ζ uniformly on any compact set of \mathcal{MF} . This means that

$$\text{MC}_{GM} \ni \zeta \mapsto \mathcal{E}xt_{ty}^{x_0}(\zeta)$$

is continuous for $ty \in \text{M}\mathcal{T}_{GM}$. The distortion property (5.7) implies that the function (5.8) is continuous. \square \square

5.3. Extremal length is intrinsic. From (2) of Lemma 5.1, the extremal length given in (5.4) is an extended notion of the usual extremal length for measured foliations. Indeed, the extremal length is *intrinsic* in the following sense.

Theorem 4 (Extremal length is intrinsic). *For $y \in \mathcal{T}_{g,m}$, there is a continuous function*

$$\text{Ext}_y(\cdot) : \mathcal{C}_{GM} \rightarrow \mathbb{R}_+$$

such that

$$(1) \quad \mathcal{E}xt_y^x(\zeta) = \text{Ext}_y \circ \tilde{\Psi}_x(\zeta) \text{ for } \zeta \in \text{MC}_{GM} \text{ and } x \in \mathcal{T}_{g,m}, \text{ and}$$

- (2) For $F \in \mathcal{MF} \subset \mathcal{C}_{GM} \subset \mathbb{R}_+^S$, the value $\text{Ext}_y(F)$ is equal to the original extremal length of F which is defined by the extension of the extremal length (2.4) on \mathcal{S} .

Remark 5.1. From the property (2) in Theorem 4, the extremal length obtained in Theorem 4 is a continuous extension of the original extremal length on \mathcal{MF} . Thus, the author believes that no confusion occurs when we use the same symbol to denote the extension of the extremal length in Theorem 4.

Proof of Theorem 4. We only check the existence and the property (1) because the property (2) follows from Lemma 5.1.

Let $x_1, x_2 \in \mathcal{T}_{g,m}$. Let $t, s > 0$ and $z, w \in \mathcal{T}_{g,m}$. Suppose that $\tilde{\Psi}_{x_1}(tz) = \tilde{\Psi}_{x_2}(sw)$. This means that

$$te^{-d_T(x_1, z)} \text{Ext}_z(\alpha)^{1/2} = se^{-d_T(x_2, w)} \text{Ext}_w(\alpha)^{1/2}$$

for all $\alpha \in \mathcal{S}$. From the injectivity of the Gardiner-Masur embedding (1.2) we have $z = w$ (cf. Lemma 6.1 in [13]). Hence

$$(5.9) \quad t = s \exp(d_T(x_1, z) - d_T(x_2, z)).$$

By Lemma 5.1, we obtain

$$\begin{aligned} \mathcal{E}xt_y^{x_1}(tz) &= t^2 \exp(-2d_T(x_1, z) + 2d_T(y, z)) \\ &= s^2 \exp(2d_T(x_1, z) - 2d_T(x_2, z)) \cdot \exp(-2d_T(x_1, z) + 2d_T(y, z)) \\ &= s^2 \exp(-2d_T(x_2, z) + 2d_T(y, z)) \\ &= \mathcal{E}xt_y^{x_2}(sz) = \mathcal{E}xt_y^{x_2}(sw). \end{aligned}$$

Therefore, there is a function $\text{Ext}_y : \mathcal{T}_{GM} \rightarrow \mathbb{R}$ such that

$$(5.10) \quad \text{Ext}_y(\mathbf{a}) = \mathcal{E}xt_y^{x_0} \circ (\tilde{\Psi}_{x_0})^{-1}(\mathbf{a}).$$

for all $\mathbf{a} \in \mathcal{T}_{GM}$. From the continuity of $\mathcal{E}xt_y^{x_0}$ on \mathcal{MC}_{GM} and $\tilde{\Psi}_{x_0}^{-1}$ on \mathcal{C}_{GM} , the function Ext_y in (5.10) extends to whole \mathcal{C}_{GM} and Equation (5.10) holds for all $\mathbf{a} \in \mathcal{C}_{GM}$. □

6. TOPOLOGY OF \mathcal{MC}_{GM}

Notice that $\text{cl}_{GM}(\mathcal{T}_{g,m})$ is metrizable (cf. [33]). Since $\mathcal{T}_{g,m}$ is separable, so is $\text{cl}_{GM}(\mathcal{T}_{g,m})$. Since $\text{cl}_{GM}(\mathcal{T}_{g,m})$ and \mathbb{R}_+ are locally compact, from the identification (4.6), \mathcal{MC}_{GM} and \mathcal{C}_{GM} are locally compact, separable and metrizable.

6.1. Bounded sets are precompact. We shall begin with the following proposition.

Proposition 6.1 (Boundedness implies compactness). *For any $R > 0$,*

$$\mathcal{MC}_{GM}(R) = \{\zeta \in \mathcal{MC}_{GM} \mid \mathcal{E}xt_{x_0}^{x_0}(\zeta) \leq R\}$$

is a compact set in \mathcal{MC}_{GM} . Furthermore, the level set

$$\{\zeta \in \mathcal{MC}_{GM} \mid \mathcal{E}xt_{x_0}^{x_0}(\zeta) = 1\}$$

coincides with $\text{cl}_{GM}(\mathcal{T}_{g,m}) \times \{1\}$. In particular $\mathcal{E}xt_{x_0}^{x_0}(\zeta) = 1$ for $\zeta \in \mathcal{MF}_1$.

Proof. By definition,

$$\mathcal{E}xt_{x_0}^{x_0}(\zeta) = \sup_{\eta \in \mathbf{MF} - \{0\}} \frac{i_{x_0}(\zeta, \eta)^2}{\text{Ext}_{x_0}(\tilde{\Psi}_{x_0}(\eta))} = \sup_{\alpha \in \mathcal{S}} \frac{i_{x_0}(\zeta, \tilde{\Psi}_{x_0}^{-1}(\alpha))^2}{\text{Ext}_{x_0}(\alpha)}.$$

Hence, the condition $\mathcal{E}xt_{x_0}^{x_0}(\zeta) \leq R$ implies that

$$i_{x_0}(\zeta, \tilde{\Psi}_{x_0}^{-1}(\alpha)) \leq R^{1/2} \text{Ext}_{x_0}(\alpha)^{1/2}$$

for all $\alpha \in \mathcal{S}$. By Tikhonov's theorem, the product of closed intervals

$$\prod_{\alpha \in \mathcal{S}} [0, R^{1/2} \text{Ext}_{x_0}(\alpha)^{1/2}]$$

is compact in $\mathbb{R}_+^{\mathcal{S}}$. From (5.3), the image of $\mathcal{MC}_{GM}(R)$ by $\tilde{\Psi}_{x_0}$ is contained in the above product. Thus, by Lemma 5.3, $\mathcal{MC}_{GM}(R)$ is closed and hence compact.

We next discuss the second claim. From (1) of Lemma 5.1,

$$\mathcal{E}xt_{x_0}^{x_0}(y) = \exp(-2d_T(x_0, y) + 2d_T(x_0, y)) = 1.$$

for $y \in \mathcal{T}_{g,m}$. From the continuity of extremal length, the image of $\text{cl}_{GM}(\mathcal{T}_{g,m})$ is contained in the level set.

Let $\zeta \in \mathcal{MC}_{GM}$ with $\mathcal{E}xt_{x_0}^{x_0}(\zeta) = 1$. By Lemma 5.2, $\zeta \neq 0$. Take $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$ and $t > 0$ such that $\zeta = tp$. Notice that $i_{x_0}(\zeta, \eta) = t \cdot i_{x_0}(p, \eta)$ holds for all $\eta \in \mathbf{MF}$.

Since

$$1 = \mathcal{E}xt_{x_0}^{x_0}(\zeta) = \mathcal{E}xt_{x_0}^{x_0}(tp) = t^2 \mathcal{E}xt_{x_0}^{x_0}(p) = t^2,$$

we have $i_{x_0}(\zeta, \eta) = i_{x_0}(p, \eta)$ for all $\eta \in \mathbf{MF}$ and hence $\zeta = p$. This means that $\zeta \in \text{cl}_{GM}(\mathcal{T}_{g,m})$. \square

6.2. A system of neighborhoods. Let $\zeta \in \mathcal{MC}_{GM} - \{0\}$, $\xi \in \mathbf{MF}$ and $\delta > 0$. We define

$$U_\delta(\zeta : \xi) = \{\eta \in \mathcal{MC}_{GM} \mid |i_{x_0}(\eta, \xi) - i_{x_0}(\zeta, \xi)| < \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \mathcal{E}xt_{x_0}^{x_0}(\xi)^{1/2} \delta\}$$

$$U_\delta(0 : \xi) = \{\eta \in \mathcal{MC}_{GM} \mid i_{x_0}(\eta, \xi) < \mathcal{E}xt_{x_0}^{x_0}(\xi)^{1/2} \delta\}.$$

Notice that

$$U_\delta(\zeta : t\xi) = U_\delta(\zeta : \xi)$$

for $t > 0$ and $\zeta, \xi \in \mathbf{MF}$ with $\xi \neq 0$. We define

$$U_\delta(\zeta) = \bigcap_{\xi \in \mathbf{MF} - \{0\}} U_\delta(\zeta : \xi).$$

We start with the following lemma.

Proposition 6.2. *Let $\delta > 0$ and $\zeta \in \mathcal{MC}_{GM}$. Then*

$$(1 - \delta) \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} < \mathcal{E}xt_{x_0}^{x_0}(\eta)^{1/2} < (1 + \delta) \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}$$

for $\eta \in U_\delta(\zeta)$.

Proof. From (5.6) and Proposition 6.1, we can find $\xi \in \mathbf{MF}_1$ such that

$$i_{x_0}(\zeta, \xi)^2 = \mathcal{E}xt_{x_0}^{x_0}(\zeta) \cdot \mathcal{E}xt_{x_0}^{x_0}(\xi) = \mathcal{E}xt_{x_0}^{x_0}(\zeta).$$

By (5.5), for $\eta \in U_\delta(\zeta)$, we have

$$\mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} = i_{x_0}(\zeta, \xi) < i_{x_0}(\eta, \xi) + \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \delta \leq \mathcal{E}xt_{x_0}^{x_0}(\eta)^{1/2} + \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \delta$$

and hence

$$(1 - \delta) \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \leq \mathcal{E}xt_{x_0}^{x_0}(\eta)^{1/2}.$$

Similarly, we take $\xi \in \mathbf{MF}_1$ with $i(\eta, \xi)^2 = \mathcal{E}xt_{x_0}^{x_0}(\eta)\mathcal{E}xt_{x_0}^{x_0}(\xi) = \mathcal{E}xt_{x_0}^{x_0}(\eta)$. This means that

$$\mathcal{E}xt_{x_0}^{x_0}(\eta)^{1/2} = i_{x_0}(\eta, \xi) < i_{x_0}(\zeta, \xi) + \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}\delta \leq \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} + \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}\delta,$$

and we are done. \square \square

We claim the following (compare Lemma 4.1 of [33]. See also [19]).

Lemma 6.1. *Let $\zeta \in \mathbf{MC}_{GM}$. For any $\delta > 0$, $U_\delta(\zeta)$ is an open neighborhood of ζ with compact closure. Furthermore, we have that $\cap_{\delta>0} U_\delta(\zeta) = \{\zeta\}$.*

Proof. It is clear that $\zeta \in U_\delta(\zeta)$ for all $\delta > 0$. To check that $U_\delta(\zeta)$ is open, we suppose on the contrary that there is a sequence $\{\zeta_n\}_{n=1}^\infty$ in the complement $\mathbf{MC}_{GM} \setminus U_\delta(\zeta)$ which converges to ζ . For any n , there is $\xi_n \in \mathbf{MF}_1$ such that

$$(6.1) \quad |i_{x_0}(\zeta_n, \xi_n) - i_{x_0}(\zeta, \xi_n)| \geq \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \mathcal{E}xt_{x_0}^{x_0}(\xi_n)^{1/2} \delta = \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \delta.$$

Since \mathbf{MF}_1 is compact, we may assume that ξ_n converges to $\xi_\infty \in \mathbf{MF}_1$. Since $\zeta_n \rightarrow \zeta$ as $n \rightarrow \infty$, by Proposition 3.1, $i_{x_0}(\zeta_n, \cdot) = \mathcal{E}_{\zeta_n} \circ \tilde{\Psi}_{x_0}(\cdot)$ converges to $i_{x_0}(\zeta_\infty, \cdot) = \mathcal{E}_{\zeta_\infty} \circ \tilde{\Psi}_{x_0}(\cdot)$ uniformly on any compact set of \mathbf{MF} . From (6.1), we have

$$0 = |i_{x_0}(\zeta, \xi_\infty) - i_{x_0}(\zeta, \xi_\infty)| \geq \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \delta > 0,$$

and we get a contradiction by Lemma 5.2. Hence $U_\delta(\zeta)$ is open. By Lemma 6.2, $U_\delta(\zeta)$ is contained in $\mathbf{MC}_{GM}((1+\delta)\mathcal{E}xt_{x_0}^{x_0}(\zeta))$. Therefore, by Proposition 6.1, the closure of $U_\delta(\zeta)$ is compact.

To show the remaining claim, we only treat the case $\zeta \neq 0$. The other case is dealt with the same manner. Suppose that $\eta \in U_\delta(\zeta)$ for all $\delta > 0$. By definition, we have

$$|i_{x_0}(\eta, \xi) - i_{x_0}(\zeta, \xi)| < \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2} \delta$$

for all $\xi \in \mathbf{MF}_1$ and $\delta > 0$. This means that $i_{x_0}(\eta, \xi) = i_{x_0}(\zeta, \xi)$ for all $\xi \in \mathbf{MF}_1$ and $\eta = \zeta$. \square \square

7. THE GROMOV PRODUCT AND AN EXTENSION OF \mathcal{E}_ζ

For $\eta = ty \in \mathbf{MT}_{GM}$ and $\zeta \in \mathbf{MC}_{GM}$, we define

$$(7.1) \quad \mathcal{E}_\eta(\zeta) = \left\{ \frac{\mathcal{E}xt_{ty}^{x_0}(\zeta)}{K_y} \right\}^{1/2} = t \cdot \exp(-d_T(x_0, y)) \cdot \mathcal{E}xt_y^{x_0}(\zeta)$$

After identifying \mathbf{MF} and \mathcal{MF} via $\tilde{\Psi}_{x_0}$, by Lemma 5.1, the function \mathcal{E}_y in (7.1) is recognized as an extension of the function (3.1) to \mathbf{MC}_{GM} . By definition, the function (7.1) satisfies the *homogeneous property*

$$(7.2) \quad \mathcal{E}_{sy}(t\zeta) = st \cdot \left\{ \frac{\mathcal{E}xt_y^{x_0}(\zeta)}{K_y} \right\}^{1/2} = st \cdot \mathcal{E}_y(\zeta).$$

for $sy \in \mathbf{MT}_{GM}$, $t \geq 0$ and $\zeta \in \mathbf{MC}_{GM}$.

7.1. The Gromov product for d_T . Notice from Lemma 5.1 that

$$(7.3) \quad \begin{aligned} \mathcal{E}_{sy}(tz) &= st \cdot \exp(-d_T(x_0, z) + d_T(y, z) - d_T(x_0, y)) \\ &= st \cdot \exp(-2\langle y | z \rangle_{x_0}) \end{aligned}$$

for $sy, tz \in \mathcal{MT}_{GM}$ where $\langle y | z \rangle_{x_0}$ is the *Gromov product*

$$\langle y | z \rangle_{x_0} = \frac{1}{2}(d_T(x_0, z) + d_T(x_0, y) - d_T(y, z))$$

with basepoint x_0 . In particular, we have the following *symmetry*

$$(7.4) \quad \mathcal{E}_{sy}(tz) = \mathcal{E}_{tz}(sy)$$

for $sy, tz \in \mathcal{MT}_{GM}$.

7.2. Equicontinuity. The following was observed for the extremal length function on \mathcal{MF} in [33].

Proposition 7.1 (Equicontinuity). *The family $\{\mathcal{E}_y\}_{y \in \mathcal{T}_{g,m}}$ is an equicontinuous family of continuous functions on \mathcal{MC}_{GM} . In fact, for $\delta > 0$ and $\zeta \in \mathcal{MC}_{GM}$, we have*

$$(7.5) \quad |\mathcal{E}_y(\zeta) - \mathcal{E}_y(\eta)| \leq \max\{1, \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}\}\delta$$

for all $\eta \in U_\delta(\zeta)$ and $y \in \mathcal{T}_{g,m}$.

Proof. We first assume that $\zeta \neq 0$. There is $\xi \in \mathcal{MF}_1$ such that $i_{x_0}(\xi, \zeta) = \mathcal{E}xt_y^{x_0}(\xi)^{1/2} \mathcal{E}xt_y^{x_0}(\zeta)^{1/2}$ (cf. (5.6)). Therefore, if $\eta \in U_\delta(\zeta)$,

$$\begin{aligned} \mathcal{E}xt_y^{x_0}(\xi)^{1/2} \mathcal{E}xt_y^{x_0}(\zeta)^{1/2} &= i_{x_0}(\xi, \zeta) \leq i_{x_0}(\xi, \eta) + \mathcal{E}xt_{x_0}^{x_0}(\zeta)\delta \\ &\leq \mathcal{E}xt_y^{x_0}(\xi)^{1/2} \mathcal{E}xt_y^{x_0}(\eta)^{1/2} + \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}\delta. \end{aligned}$$

Hence we get

$$(7.6) \quad \begin{aligned} \mathcal{E}xt_y^{x_0}(\zeta)^{1/2} &\leq \mathcal{E}xt_y^{x_0}(\eta)^{1/2} + \frac{\mathcal{E}xt_{x_0}^{x_0}(\zeta)}{\mathcal{E}xt_y^{x_0}(\xi)}\delta \\ &\leq \mathcal{E}xt_y^{x_0}(\eta)^{1/2} + K_y^{1/2} \mathcal{E}xt_{x_0}^{x_0}(\zeta)\delta, \end{aligned}$$

since $\mathcal{E}xt_y^{x_0}(\xi) \geq K_y^{-1} \mathcal{E}xt_{x_0}^{x_0}(\xi) = K_y^{-1}$ (cf. (5.7)).

We also take $\xi' \in \mathcal{MF}_1$ with $i_{x_0}(\xi', \eta) = \mathcal{E}xt_y^{x_0}(\xi')^{1/2} \mathcal{E}xt_y^{x_0}(\eta)^{1/2}$. Then,

$$\begin{aligned} \mathcal{E}xt_y^{x_0}(\xi')^{1/2} \mathcal{E}xt_y^{x_0}(\eta)^{1/2} &= i_{x_0}(\xi', \eta) \leq i_{x_0}(\xi', \zeta) + \mathcal{E}xt_{x_0}^{x_0}(\zeta)\delta \\ &\leq \mathcal{E}xt_y^{x_0}(\xi')^{1/2} \mathcal{E}xt_y^{x_0}(\zeta)^{1/2} + \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}\delta. \end{aligned}$$

Hence, by the same argument as above,

$$(7.7) \quad \mathcal{E}xt_y^{x_0}(\eta)^{1/2} \leq \mathcal{E}xt_y^{x_0}(\zeta)^{1/2} + K_y^{1/2} \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}\delta.$$

Thus, (7.6) and (7.7)

$$(7.8) \quad |\mathcal{E}xt_y^{x_0}(\eta)^{1/2} - \mathcal{E}xt_y^{x_0}(\zeta)^{1/2}| \leq K_y^{1/2} \mathcal{E}xt_{x_0}^{x_0}(\zeta)^{1/2}\delta.$$

Suppose $\zeta = 0$. If we take $\xi' \in \mathcal{MF}_1$ with $i_{x_0}(\eta, \xi') = \mathcal{E}xt_y^{x_0}(\xi')^{1/2} \mathcal{E}xt_y^{x_0}(\eta)^{1/2}$, then,

$$\mathcal{E}xt_y^{x_0}(\xi')^{1/2} \cdot \mathcal{E}xt_y^{x_0}(\eta)^{1/2} = i_{x_0}(\eta, \xi') < \delta.$$

Therefore, we conclude

$$(7.9) \quad |\mathcal{E}xt_y^{x_0}(\eta)^{1/2} - \mathcal{E}xt_y^{x_0}(\zeta)^{1/2}| = \mathcal{E}xt_y^{x_0}(\eta) \leq \frac{\delta}{\mathcal{E}xt_y^{x_0}(\xi')^{1/2}} \leq K_y^{1/2}\delta.$$

Thus, (7.8) and (7.9) implies (7.5). \square \square

8. AN EXTENSION OF THE INTERSECTION NUMBER

The purpose of this section is to show the following theorem.

Theorem 5 (Intersection number on \mathcal{C}_{GM}). *There exists a unique continuous function*

$$(8.1) \quad i(\cdot, \cdot) : \mathcal{C}_{GM} \times \mathcal{C}_{GM} \rightarrow \mathbb{R}_+$$

independent of the choice of basepoint x_0 satisfying the following properties.

- (1) *For any $\zeta, \eta \in \mathcal{MC}_{GM}$,*

$$i(\tilde{\Psi}_{x_0}(\zeta), \tilde{\Psi}_{x_0}(\eta)) = i_{x_0}(\zeta, \eta).$$

In particular, we have

$$\begin{aligned} i(\tilde{\Psi}_{x_0}(ty), \tilde{\Psi}_{x_0}(sp)) &= ts e^{-d_T(x_0, y)} \text{Ext}_y(\Psi_{x_0}(p))^{1/2} \\ i(\tilde{\Psi}_{x_0}(p), F) &= i(\Psi_{x_0}(p), F) = \mathcal{E}_p(F) \end{aligned}$$

for $x_0, y \in \mathcal{T}_{g,m}$, $p \in \partial_{GM}\mathcal{T}_{g,m}$, $F \in \mathcal{MF}$ and $t, s \geq 0$.

- (2) *$i(\mathbf{a}, \mathbf{b}) = i(\mathbf{b}, \mathbf{a})$ for $\mathbf{a}, \mathbf{b} \in \mathcal{C}_{GM}$.*
(3) *$i(s\mathbf{a}, t\mathbf{b}) = st \cdot i(\mathbf{a}, \mathbf{b})$ for $s, t \geq 0$ and $\mathbf{a}, \mathbf{b} \in \mathcal{C}_{GM}$,*
(4) *For $x_0, y, z \in \mathcal{T}_{g,m}$,*

$$i(\Psi_{x_0}(y), \Psi_{x_0}(z)) = \exp(-2\langle y | z \rangle_{x_0}).$$

- (5) *The self-intersection number satisfies*

$$i(\mathbf{a}, \mathbf{a}) = \begin{cases} t^2 \exp(-2d_T(x_0, y)) & \text{if } \mathbf{a} = \tilde{\Psi}_{x_0}(ty) \in \mathcal{T}_{GM} \\ 0 & \text{if } \mathbf{a} \in \partial_{GM}. \end{cases}$$

for $x_0 \in \mathcal{T}_{g,m}$.

- (6) *For $F, G \in \mathcal{MF} \subset \mathcal{C}_{GM}$,*

$$i(F, G) = I(F, G)$$

where we recall that the intersection number in the right-hand side is the original intersection number function on $\mathcal{MF} \times \mathcal{MF}$.

Corollaries. Before proving Theorem 5, we give two corollaries as follows. The first corollary tells us that the formula (1.9) holds for arbitrary elements in \mathcal{C}_{GM} .

Corollary 3 (Intrinsic representation of extremal length). *For $y \in \mathcal{T}_{g,m}$ and $\mathbf{a} \in \mathcal{C}_{GM}$, we have*

$$(8.2) \quad \text{Ext}_y(\mathbf{a}) = \sup_{F \in \mathcal{MF} - \{0\}} \frac{i(\mathbf{a}, F)^2}{\text{Ext}_y(F)}.$$

Proof. Notice that in the definition (5.4) of the extremal length, the measured foliation F in the numerator in (5.4) is taken in $\mathcal{MF} - \{0\} \subset \mathcal{MC}_{GM}$. Therefore, by

Theorem 5, for $\mathbf{a} \in \mathcal{C}_{GM}$, we have

$$\begin{aligned} \text{Ext}_y(\mathbf{a}) &= \mathcal{E}xt_y^{x_0} \circ \tilde{\Psi}_{x_0}^{-1}(\mathbf{a}) \\ &= \sup_{F \in \mathcal{MF} - \{0\}} \frac{i_{x_0}(\tilde{\Psi}_{x_0}^{-1}(\mathbf{a}), \tilde{\Psi}_{x_0}^{-1}(F))^2}{\text{Ext}_y(F)} \\ &= \sup_{F \in \mathcal{MF} - \{0\}} \frac{i(\mathbf{a}, F)^2}{\text{Ext}_y(F)}, \end{aligned}$$

where F in the supremum above runs over all $F \in \mathcal{MF} - \{0\} \subset \mathcal{C}_{GM}$, and the extremal length in the denominator is defined in Theorem 4. \square \square

The second corollary tells us that our intersection number has an expected property. Namely, Minsky's inequality (2.6) holds for elements of \mathcal{C}_{GM} .

Corollary 4 (Minsky's inequality). *For $x \in \mathcal{T}_{g,m}$ and $\mathbf{a}, \mathbf{b} \in \mathcal{C}_{GM}$, we have*

$$(8.3) \quad i(\mathbf{a}, \mathbf{b})^2 \leq \text{Ext}_x(\mathbf{a}) \text{Ext}_x(\mathbf{b}).$$

The equality holds if the projective classes of \mathbf{a}, x and \mathbf{b} are on a common Teichmüller geodesic in this order.

Proof. Suppose that $\mathbf{a}, \mathbf{b} \in \mathcal{T}_{GM}$. Take $ty, sz \in \mathcal{MT}_{GM}$ with $\mathbf{a} = \tilde{\Psi}_{x_0}(ty)$ and $\mathbf{b} = \tilde{\Psi}_{x_0}(sz)$. Then, by Lemma 5.1, we have

$$\begin{aligned} i(\mathbf{a}, \mathbf{b})^2 &= i_{x_0}(\tilde{\Psi}_{x_0}(ty), \tilde{\Psi}_{x_0}(sz))^2 = t^2 s^2 i_{x_0}(\Psi_{x_0}(y), \Psi_{x_0}(z))^2 \\ &= t^2 s^2 \exp(-4\langle y | z \rangle_{x_0}) \\ &= t^2 s^2 \exp(2d_T(y, z) - 2d_T(x_0, y) - 2d_T(x_0, z)) \\ (8.4) \quad &\leq t^2 s^2 \exp(2d_T(x, y) - 2d_T(x_0, y)) \cdot \exp(2d_T(x, z) - 2d_T(x_0, z)) \\ &= t^2 s^2 \mathcal{E}xt_x^{x_0}(y) \cdot \mathcal{E}xt_x^{x_0}(z) = \mathcal{E}xt_x^{x_0}(ty) \cdot \mathcal{E}xt_x^{x_0}(sz) \\ &= \text{Ext}_x(\mathbf{a}) \cdot \text{Ext}_x(\mathbf{b}). \end{aligned}$$

Since \mathcal{T}_{GM} is dense in \mathcal{C}_{GM} , we have the desired inequality.

Suppose the projective classes of \mathbf{a}, x and \mathbf{b} are on a common Teichmüller geodesic $\gamma : \mathbb{R} \rightarrow \mathcal{T}_{g,m}$ in this order. We may assume that $\mathbf{a}, \mathbf{b} \in \partial_{GM} \mathcal{T}_{g,m}$ since intersection number and extremal length are homogeneous. The other cases can be treated in the same manner. From the assumption, we may choose γ such that $\gamma(t) \rightarrow \mathbf{a}$ and $\gamma(-t) \rightarrow \mathbf{b}$ when $t \rightarrow \infty$. Therefore, from (8.4) we have

$$i(\gamma(t), \gamma(-t))^2 = \text{Ext}_x(\gamma(t)) \cdot \text{Ext}_x(\gamma(-t))$$

for sufficiently large $t > 0$. By letting $t \rightarrow \infty$, we get the equality in (8.3). \square \square

8.1. Extension of the intersection number i_{x_0} . To show Theorem 5, we first extend the intersection number (5.1) to the whole $\mathcal{MC}_{GM} \times \mathcal{MC}_{GM}$.

Proposition 8.1 (Extension of i_{x_0}). *For any $x_0 \in \mathcal{T}_{g,m}$, there exists a unique continuous function*

$$(8.5) \quad i_{x_0}(\cdot, \cdot) : \mathcal{MC}_{GM} \times \mathcal{MC}_{GM} \rightarrow \mathbb{R}_+$$

such that

$$\begin{aligned} (1) \quad &\text{For } ty \in \mathcal{MT}_{GM} \text{ and } sp \in \mathcal{M}\tilde{\partial}_{GM} \text{ with } y \in \mathcal{T}_{g,m}, p \in \partial_{GM} \mathcal{T}_{g,m} \text{ and } t, s \geq 0, \\ &i_{x_0}(ty, sp) = ts \mathcal{E}_y(p) = ts e^{-d_T(x_0, y)} \mathcal{E}xt_y^{x_0}(p)^{1/2}, \end{aligned}$$

- (2) $i_{x_0}(\zeta, \eta) = i_{x_0}(\eta, \zeta)$ for $\zeta, \eta \in \mathbf{MC}_{GM}$;
- (3) $i_{x_0}(s\zeta, t\eta) = st \cdot i_{x_0}(\zeta, \eta)$ for $s, t \geq 0$ and $\zeta, \eta \in \mathbf{MC}_{GM}$;
- (4) for $y, z \in \mathcal{T}_{g,m}$, $i_{x_0}(y, z) = \exp(-2\langle y | z \rangle_{x_0})$;
- (5) for $\zeta = tp \in \mathbf{MC}_{GM}$ with $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$;

$$i_{x_0}(\zeta, \zeta) = \begin{cases} t^2 \exp(-2d_T(x_0, p)) & \text{if } \zeta \in \mathbf{MT}_{GM} \\ 0 & \text{if } \zeta \in \mathbf{M}\tilde{\partial}_{GM}; \end{cases}$$

- (6) $i_{x_0}(\tilde{\Psi}_{x_0}^{-1}(F), \tilde{\Psi}_{x_0}^{-1}(G)) = I(F, G)$ for all $F, G \in \mathcal{MF}$.

Proof. Consider an equicontinuous family $\{\mathcal{E}_y\}_{y \in \mathcal{T}_{g,m}}$ given in Proposition 7.1. For any $\zeta \in \mathbf{MC}_{GM}$,

$$\mathcal{E}_y(\zeta) = \left\{ \frac{\mathcal{E}xt_y^{x_0}(\zeta)}{K_y} \right\}^{1/2} \leq \mathcal{E}xt_{x_0}^{x_0}(\zeta).$$

By Proposition 6.1, the family $\{\mathcal{E}_y\}_{y \in \mathcal{T}_{g,m}}$ is uniformly bounded on any compact set. Therefore, the family is a normal family.

Let $\zeta \in \mathbf{MC}_{GM}$. Let $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$ and $t > 0$ such that $\zeta = tp$. Let $\{y_n\}_{n=1}^\infty$ be a sequence converging to p . Take a sequence $\{t_n\}_{n=1}^\infty$ of positive numbers with $t_n \rightarrow t$. By Ascoli-Arzelà theorem, there is a subsequence $\{y_{n_j}\}_j$ such that a sequence $\{\mathcal{E}_{t_{n_j}y_{n_j}}\}_j$ converges to a continuous function \mathcal{E}' on \mathbf{MC}_{GM} uniformly on any compact set. Notice that for $sz \in \mathbf{MT}_{GM}$, from Lemma 5.3 and (7.4),

$$(8.6) \quad \mathcal{E}'(sz) = \lim_{j \rightarrow \infty} \mathcal{E}_{t_{n_j}y_{n_j}}(sz) = \lim_{j \rightarrow \infty} \mathcal{E}_{sz}(t_{n_j}y_{n_j}) = \mathcal{E}_{sz}(\zeta).$$

Take another sequence $\{t'_k y'_k\}_k$ in \mathbf{MT}_{GM} which tends to ζ such that $\mathcal{E}_{t_k y'_k}$ converges to a continuous function \mathcal{E}'' on \mathbf{MC}_{GM} uniformly on any compact set of \mathbf{MC}_{GM} . Since the right-hand side of (8.6) is independent of converging sequences, the same conclusion holds for \mathcal{E}'' . Namely, we have

$$\mathcal{E}''(sz) = \mathcal{E}_{sz}(\zeta) = \mathcal{E}'(sz)$$

for all $sz \in \mathbf{MT}_{GM}$. Since \mathbf{MT}_{GM} is dense in \mathbf{MC}_{GM} and both \mathcal{E}'' and \mathcal{E}' are continuous on \mathbf{MC}_{GM} , $\mathcal{E}'' = \mathcal{E}'$ on \mathbf{MC}_{GM} . This means that the limit \mathcal{E}' above is dependent only on ζ , independent of the choice of the sequence $\{y_n\}_{n=1}^\infty$ converging to ζ . We denote by $i_{x_0}(\zeta, \cdot)$ the limit.

For any $R > 0$, notice again that $\{\mathcal{E}_{sy}\}_{sy \in \mathbf{MC}_{GM}(R)}$ is a normal family of continuous functions on \mathbf{MC}_{GM} . Hence, from the argument above,

$$(8.7) \quad \mathbf{MC}_{GM} \times \mathbf{MC}_{GM} \ni (\zeta, \eta) \mapsto i_{x_0}(\zeta, \eta)$$

is continuous in two variables. The condition (1) in the statement follows from the construction and (7.1). From the density of $\mathbf{MT}_{GM} \times \mathbf{MC}_{GM}$ in $\mathbf{MC}_{GM} \times \mathbf{MC}_{GM}$ we deduce the uniqueness of our function $i_{x_0}(\cdot, \cdot)$.

Let us check that our function $i_{x_0}(\cdot, \cdot)$ satisfies the remaining conditions (2) to (5) in the statement. Indeed, (2) and (3) follows from the density of \mathbf{MT}_{GM} in \mathbf{MC}_{GM} and equations (7.2) and (7.4). We get (4) from (7.3). The condition (5) is verified from

$$i_{x_0}(\zeta, \zeta) = t^2 \exp(-2\langle y | y \rangle_{x_0}) = t^2 \exp(-d_T(x_0, y))$$

when $\zeta = ty \in \mathbf{MT}_{GM}$ and the continuity of the function $i_{x_0}(\cdot, \cdot)$. The last condition (6) follows from Proposition 5.1. \square \square

8.2. Intersection number is intrinsic. We define

$$(8.8) \quad i(\mathfrak{a}, \mathfrak{b}) = i_{x_0}(\tilde{\Psi}_{x_0}^{-1}(\mathfrak{a}), \tilde{\Psi}_{x_0}^{-1}(\mathfrak{b}))$$

for $\mathfrak{a}, \mathfrak{b} \in \mathcal{MC}_{GM}$. It follows from (1.6) that the intersection number (8.8) is intrinsic in the sense that the value is independent of the choice of basepoint x_0 . In this section, we shall give a more direct proof.

Proposition 8.2 (Intersection number is intrinsic). *The function (8.8) is defined independently of the choice of the basepoint. Namely, for $x_1, x_2 \in \mathcal{T}_{g,m}$, we have*

$$(8.9) \quad i_{x_1}(\tilde{\Psi}_{x_1}^{-1}(\mathfrak{a}), \tilde{\Psi}_{x_1}^{-1}(\mathfrak{b})) = i_{x_2}(\tilde{\Psi}_{x_2}^{-1}(\mathfrak{a}), \tilde{\Psi}_{x_2}^{-1}(\mathfrak{b}))$$

for $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}_{GM}$.

Proof. We first notice from the definition of \mathcal{MT}_{GM} (cf. (4.2)) that the cone \mathcal{MT}_{GM} is defined independently of the basepoint. Hence, from the density of \mathcal{T}_{GM} in \mathcal{C}_{GM} , it suffices to check (8.9) only on $\mathcal{T}_{GM} \times \mathcal{T}_{GM}$.

Let $\mathfrak{a}_i \in \mathcal{T}_{GM}$. Then, there exist $t_{i,j} > 0$ ($i, j = 1, 2$) and $y_{i,j} \in \mathcal{T}_{g,m}$ such that

$$(8.10) \quad \mathfrak{a}_i = \tilde{\Psi}_{x_j}(t_{i,j}y_{i,j}) = t_{i,j} \Psi_{x_j}(y_{i,j}) = [\mathcal{S} \ni \alpha \mapsto t_{i,j}\mathcal{E}_{y_{i,j}}(\alpha)].$$

For $i = 1, 2$, the projective classes of $\Psi_{x_1}(y_{i,1})$ and $\Psi_{x_2}(y_{i,2})$ agree and hence $y_{i,1} = y_{i,2}$ by Lemma 6.1 in [13]. From now on, we let $y_i = y_{i,1} = y_{i,2}$. From (8.10),

$$t_{i,1} \frac{\text{Ext}_{y_i}(\alpha)^{1/2}}{\exp(d_T(x_1, y_i))} = \mathfrak{a}_i(\alpha) = t_{i,2} \frac{\text{Ext}_{y_i}(\alpha)^{1/2}}{\exp(d_T(x_2, y_i))}$$

for all $\alpha \in \mathcal{S}$, where $\mathfrak{a}_i(\alpha)$ is the α -coordinate of \mathfrak{a}_i for $\alpha \in \mathcal{S}$. Equivalently, we have

$$t_{i,1} \exp(-d_T(x_1, y_i)) = t_{i,2} \exp(-d_T(x_2, y_i))$$

and hence

$$(8.11) \quad t_{1,1}t_{2,1} = t_{1,2}t_{2,2} \frac{\exp(d_T(x_1, y_1) + d_T(x_1, y_2))}{\exp(d_T(x_2, y_1) + d_T(x_2, y_2))}.$$

Thus, by (8.11) and (4) of Proposition 8.1, we have

$$\begin{aligned} i_{x_1}(\tilde{\Psi}_{x_1}^{-1}(\mathfrak{a}_1), \tilde{\Psi}_{x_1}^{-1}(\mathfrak{a}_2)) &= i_{x_1}(t_{1,1}y_1, t_{2,1}y_2) \\ &= t_{1,1}t_{2,1} \cdot i_{x_1}(y_1, y_2) \\ &= t_{1,1}t_{2,1} \cdot \exp(-2\langle y_1 | y_2 \rangle_{x_1}) \\ &= t_{1,2}t_{2,2} \frac{\exp(d_T(x_1, y_1) + d_T(x_1, y_2))}{\exp(d_T(x_2, y_1) + d_T(x_2, y_2))} \\ &\quad \times \exp(-d_T(x_1, y_1) - d_T(x_1, y_2) + d_T(y_1, y_2)) \\ &= t_{1,2}t_{2,2} \cdot i_{x_2}(y_1, y_2) \\ &= i_{x_2}(\tilde{\Psi}_{x_2}^{-1}(\mathfrak{a}_1), \tilde{\Psi}_{x_2}^{-1}(\mathfrak{a}_2)), \end{aligned}$$

which is what we wanted. \square

\square

8.3. Extension of the Gromov product. In this section, we give a proof of Corollary 1 which asserts that the Gromov product admits a unique continuous extension to $\text{cl}_{GM}(\mathcal{T}_{g,m})$.

Proof of Corollary 1. The uniqueness of the extension follows from the density of $\mathcal{T}_{g,m}$ in $\text{cl}_{GM}(\mathcal{T}_{g,m})$ and the condition (1) in the assertion of Corollary 1. Hence it suffices to show existence.

Define

$$(8.12) \quad \langle \zeta | \eta \rangle_{x_0} = -\frac{1}{2} \log i_{x_0}(\zeta, \eta)$$

for $\zeta, \eta \in \text{cl}_{GM}(\mathcal{T}_{g,m})$, where $\text{cl}_{GM}(\mathcal{T}_{g,m})$ is identified with a subset via the embedding (4.5). Notice from Proposition 6.1 and Corollary 4 that $i_{x_0}(\zeta, \eta) \leq 1$ for $\zeta, \eta \in \text{cl}_{GM}(\mathcal{T}_{g,m})$. Hence, by Proposition 8.1, $i_{x_0}(\cdot, \cdot)$ is a non-negative continuous function on $\text{cl}_{GM}(\mathcal{T}_{g,m}) \times \text{cl}_{GM}(\mathcal{T}_{g,m})$. Therefore, the pairing $\langle \cdot | \cdot \rangle_{x_0}$ defined above is continuous with value in $[0, \infty]$. From (4) of Proposition 8.1, the pairing in (8.12) satisfies the condition (1) in the assertion.

Since

$$\begin{aligned} I(F, G) &= i_{x_0}(\tilde{\Psi}_{x_0}^{-1}(F), \tilde{\Psi}_{x_0}^{-1}(G)) \\ &= i_{x_0}(\text{Ext}_{x_0}(F)^{1/2}[F], \text{Ext}_{x_0}(G)^{1/2}[G]) \\ &= \text{Ext}_{x_0}(F)^{1/2} \cdot \text{Ext}_{x_0}(G)^{1/2} i_{x_0}([F], [G]), \end{aligned}$$

we have

$$\exp(-2\langle [F] | [G] \rangle_{x_0}) = i_{x_0}([F], [G]) = \frac{I(F, G)}{\text{Ext}_{x_0}(F)^{1/2} \cdot \text{Ext}_{x_0}(G)^{1/2}},$$

which is what we wanted. \square \square

9. ISOMETRIC ACTION OF $\mathcal{T}_{g,m}$

An orientation preserving homeomorphism $h: X \rightarrow X$ induces a homeomorphic action h_* on $\partial_{GM}\mathcal{T}_{g,m}$ by the equation

$$(9.1) \quad \mathcal{E}_{h_*(p)}(F) = t \mathcal{E}_p(h^{-1}(F))$$

for all $F \in \mathcal{MF}$, where $t > 0$ is independent of F . Indeed, the action h_* is the homeomorphic extension of the Teichmüller modular group action on $\mathcal{T}_{g,m}$ induced by h (cf. §5.4 of [33]). In this section, we give a necessary condition for a mapping of $\partial_{GM}\mathcal{T}_{g,m}$ to be induced from a homeomorphism on X .

9.1. Mapping of bounded distortion for triangles. Recall that a mapping $\omega: \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ is said to be a *mapping of bounded distortion for triangles with distortion constants D_1 and D_2* if

$$\frac{1}{D_1} \langle x | y \rangle_z - D_2 \leq \langle \omega(x) | \omega(y) \rangle_{\omega(z)} \leq D_1 \langle x | y \rangle_z + D_2$$

for all $x, y \in \mathcal{T}_{g,m}$ (cf. §1.3.6). A mapping $\omega': \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ is said to be a *quasi-inverse* of $\omega: \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ if there is a constant $D_3 > 0$ such that

$$\sup_{x \in \mathcal{T}_{g,m}} \{d_T(x, \omega \circ \omega'(x)), d_T(x, \omega' \circ \omega(x))\} \leq D_3.$$

We note the following simple lemma.

Lemma 9.1 (Quasi-inverse). *Let ω be a mapping of bounded distortion for triangles. Then, any quasi-inverse ω' of ω is also a mapping of bounded distortion for triangles.*

Proof. Let $x, y, z \in \mathcal{T}_{g,m}$. A simple calculation shows

$$\begin{aligned} \langle \omega'(x) | \omega'(y) \rangle_{\omega'(z)} &\leq D_1 \langle \omega \circ \omega'(x) | \omega \circ \omega'(y) \rangle_{\omega \circ \omega'(z)} + D_1 D_2 \\ &\leq D_1 (\langle x | y \rangle_z + 3D_3) + D_1 D_2 \\ &= D_1 \langle x | y \rangle_z + D_1 (D_2 + 3D_3). \end{aligned}$$

The converse also holds in the same manner. \square \square

9.2. Null space. For $\mathbf{a} \in \mathcal{C}_{GM}$, we define the *null space* of \mathbf{a} by

$$\mathcal{N}(\mathbf{a}) = \{\mathbf{b} \in \mathcal{C}_{GM} \mid i(\mathbf{a}, \mathbf{b}) = 0\}.$$

By definition, $0 \in \mathcal{N}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{C}_{GM}$. We remark the following simple claim.

Proposition 9.1. *The following hold.*

- (1) For $\mathbf{a} \in \mathcal{C}_{GM}$, $\mathcal{N}(\mathbf{a}) \neq \{0\}$ if and only if $\mathbf{a} \in \tilde{\partial}_{GM}$.
- (2) $\mathcal{N}(\mathbf{a}) \subset \tilde{\partial}_{GM}$ for all $\mathbf{a} \in \mathcal{C}_{GM}$.
- (3) $\mathcal{N}(\mathbf{a}) \cap \mathcal{MF} \neq \{0\}$ for $\mathbf{a} \in \tilde{\partial}_{GM}$.

Proof. (1) If $\mathbf{a} \in \mathcal{T}_{GM}$, from Lemma 5.2 and (1) of Theorem 5, $\mathcal{N}(\mathbf{a}) = \{0\}$. If $\mathbf{a} \in \tilde{\partial}_{GM}$, from (5) of Theorem 5, we have $\mathbf{a} \in \mathcal{N}(\mathbf{a})$ and $\mathcal{N}(\mathbf{a}) \neq \{0\}$.

(2) From (1) above, $\mathcal{N}(\mathbf{a}) = \{0\} \subset \tilde{\partial}_{GM}$ for $\mathbf{a} \in \mathcal{T}_{GM}$. Let $\mathbf{a} \in \tilde{\partial}_{GM}$. For any $\mathbf{b} \in \mathcal{N}(\mathbf{a})$, $\mathbf{a} \in \mathcal{N}(\mathbf{b}) \neq \{0\}$. This means that $\mathcal{N}(\mathbf{a}) \cap \mathcal{T}_{GM} = \{0\}$ for all $\mathbf{a} \in \mathcal{C}_{GM}$.

(3) Let $\mathbf{a} \in \tilde{\partial}_{GM}$. Suppose $\mathbf{a} = \tilde{\Psi}_{GM}(tp)$ for some $t \geq 0$ and $p \in \partial_{GM}\mathcal{T}_{g,m}$. If $\mathcal{N}(\mathbf{a}) \cap \mathcal{MF} = \{0\}$,

$$t\mathcal{E}_p(F) = i(\tilde{\Psi}_{x_0}(tp), \tilde{\Psi}_{x_0}(F)) = i(\mathbf{a}, F) \neq 0$$

for all $F \in \mathcal{MF} - \{0\}$ by Theorem 5. By Proposition 3.2, this implies $p \in \mathcal{T}_{g,m}$, which is a contradiction. \square \square

Let ω be a mapping $\omega: \text{cl}_{GM}(\mathcal{T}_{g,m}) \rightarrow \text{cl}_{GM}(\mathcal{T}_{g,m})$. We extend the action of ω to MC_{GM} by

$$H_\omega: \text{MC}_{GM} \ni tp \mapsto t\omega(p) \in \text{MC}_{GM}$$

where $t \geq 0$ and $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$. Let $x_0 \in \mathcal{T}_{g,m}$ be the basepoint as before. We define a homeomorphism h_ω on \mathcal{C}_{GM} by

$$h_\omega = \tilde{\Psi}_{x_0} \circ H_\omega \circ \tilde{\Psi}_{x_0}^{-1}.$$

Proposition 9.2. *Let $\omega: \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ be a mapping of bounded distortion for triangles. Suppose that ω admits a continuous extension to $\text{cl}_{GM}(\mathcal{T}_{g,m})$. Then, $i(h_\omega(\mathbf{a}), h_\omega(\mathbf{b})) = 0$ if and only if $i(\mathbf{a}, \mathbf{b}) = 0$ for $\mathbf{a}, \mathbf{b} \in \tilde{\partial}_{GM}$. Furthermore, if ω has a quasi-inverse ω' which also admits a continuous extension to $\text{cl}_{GM}(\mathcal{T}_{g,m})$, then*

$$(9.2) \quad h_{\omega'} \circ h_\omega(\mathcal{N}(\mathbf{a})) \subset \mathcal{N}(\mathbf{a})$$

when $\mathbf{a} \in \tilde{\partial}_{GM}$.

Proof. Let D_1 and D_2 be the distortion constants of ω . We may assume that $\mathbf{a} \in \partial_{GM}$ from Proposition 9.1. Notice from a formal calculation that

$$\begin{aligned} 2\langle \omega(y) | \omega(z) \rangle_{\omega(x_0)} &= d_T(\omega(x_0), \omega(y)) + d_T(\omega(x_0), \omega(z)) - d_T(\omega(y), \omega(z)) \\ &= d_T(x_0, \omega(y)) + d_T(x_0, \omega(z)) - d_T(\omega(y), \omega(z)) \\ &\quad + (d_T(\omega(x_0), \omega(y)) - d_T(x_0, \omega(y))) \\ &\quad + (d_T(\omega(x_0), \omega(z)) - d_T(x_0, \omega(z))) \\ &= 2\langle \omega(y) | \omega(z) \rangle_{x_0} - 2\langle \omega(x_0) | \omega(y) \rangle_{x_0} \\ &\quad - 2\langle \omega(x_0) | \omega(z) \rangle_{x_0} + 2d_T(x_0, \omega(x_0)) \end{aligned}$$

for every $x, y \in \mathcal{T}_{g,m}$. Since ω is a mapping of bounded distortion for triangles with constant $D_1, D_2 > 0$,

$$\frac{1}{D_1} \langle x | y \rangle_{x_0} - D_2 \leq \langle \omega(x) | \omega(y) \rangle_{\omega(x_0)} \leq D_1 \langle y | z \rangle_{x_0} + D_2.$$

Therefore we conclude that

$$(9.3) \quad e^{-2D_2} J_{x_0}(y, z) i_{x_0}(y, z)^{D_1} \leq i_{x_0}(\omega(y), \omega(z)) \leq e^{2D_2} J_{x_0}(y, z) i_{x_0}(y, z)^{\frac{1}{D_1}},$$

where

$$J_{x_0}(y, z) = \frac{i_{x_0}(\omega(x_0), \omega(y)) i_{x_0}(\omega(x_0), \omega(z))}{e^{2d_T(x_0, \omega(x_0))}}.$$

Let $\zeta, \eta \in \partial_{GM} \mathcal{T}_{g,m}$. Since ω has a continuous extension to $\text{cl}_{GM}(\mathcal{T}_{g,m})$, by letting $y \rightarrow \zeta$ and $z \rightarrow \eta$ in (9.3), we get

$$(9.4) \quad e^{-2D_2} J_{x_0}(\zeta, \eta) i_{x_0}(\zeta, \eta)^{D_1} \leq i_{x_0}(\omega(\zeta), \omega(\eta)) \leq e^{2D_2} J_{x_0}(\zeta, \eta) i_{x_0}(\zeta, \eta)^{\frac{1}{D_1}}$$

from Proposition 8.1, where

$$\begin{aligned} J_{x_0}(\zeta, \eta) &= \lim_{y \rightarrow \zeta, z \rightarrow \eta} \frac{i_{x_0}(\omega(x_0), \omega(y)) i_{x_0}(\omega(x_0), \omega(z))}{e^{2d_T(x_0, \omega(x_0))}} \\ &= \frac{\mathcal{E} x t_{\omega(x_0)}^{x_0}(\omega(\zeta))^{1/2} \mathcal{E} x t_{\omega(x_0)}^{x_0}(\omega(\eta))^{1/2}}{e^{4d_T(x_0, \omega(x_0))}} \neq 0 \end{aligned}$$

since $\omega(x_0) \in \mathcal{T}_{g,m}$ (cf. Lemma 5.2). Therefore, (9.4) implies that $i_{x_0}(\omega(\zeta), \omega(\eta)) = 0$ if and only if $i_{x_0}(\zeta, \eta) = 0$ for $\zeta, \eta \in \partial_{GM} \mathcal{T}_{g,m}$.

Let $\mathbf{a}, \mathbf{b} \in \tilde{\partial}_{GM}$. Take $\zeta, \eta \in \partial_{GM} \mathcal{T}_{g,m}$ and $t, s \geq 0$ with $\mathbf{a} = \tilde{\Psi}_{x_0}(t\zeta)$ and $\mathbf{b} = \tilde{\Psi}_{x_0}(s\eta)$. Then, by (8.8),

$$\begin{aligned} i(\mathbf{a}, \mathbf{b}) &= i_{x_0}(t\zeta, s\eta) = ts i_{x_0}(\zeta, \eta) \\ i(h_\omega(\mathbf{a}), h_\omega(\mathbf{b})) &= i_{x_0}(H_\omega(t\zeta), H_\omega(s\eta)) = ts i_{x_0}(\omega(\zeta), \omega(\eta)). \end{aligned}$$

Therefore, $i(\mathbf{a}, \mathbf{b}) = 0$ if and only if $i(h_\omega(\mathbf{a}), h_\omega(\mathbf{b})) = 0$.

Suppose ω has a quasi-inverse ω' of quasi-inverse constant D_3 which extends continuously to $\text{cl}_{GM}(\mathcal{T}_{g,m})$. Then,

$$2\langle y | z \rangle_{x_0} - 2D_3 \leq 2\langle y | \omega' \circ \omega(z) \rangle_{x_0} \leq 2\langle y | z \rangle_{x_0} + 2D_3$$

and

$$e^{-2D_3} i_{x_0}(y, z) \leq i_{x_0}(y, \omega' \circ \omega(z)) \leq e^{2D_3} i_{x_0}(y, z).$$

Therefore, by letting $y \rightarrow \zeta$ and $z \rightarrow \eta$, we have

$$e^{-2D_3} i_{x_0}(\zeta, \eta) \leq i_{x_0}(\zeta, \omega' \circ \omega(\eta)) \leq e^{2D_3} i_{x_0}(\zeta, \eta)$$

for all $\zeta, \eta \in \partial_{GM} \mathcal{T}_{g,m}$, which implies

$$e^{-2D_3} i(\mathbf{a}, \mathbf{b}) \leq i(\mathbf{a}, h_{\omega'} \circ h_{\omega}(\mathbf{b})) \leq e^{2D_3} i(\mathbf{a}, \mathbf{b})$$

for $\mathbf{a}, \mathbf{b} \in \tilde{\partial}_{GM}$. Let $\mathbf{b} \in h_{\omega'} \circ h_{\omega}(\mathcal{N}(\mathbf{a}))$. Take $\mathbf{c} \in \mathcal{N}(\mathbf{a})$ with $\mathbf{b} = h_{\omega'} \circ h_{\omega}(\mathbf{c})$. Since

$$i(\mathbf{a}, \mathbf{b}) = i(\mathbf{a}, h_{\omega'} \circ h_{\omega}(\mathbf{c})) \leq e^{2D_3} i(\mathbf{a}, \mathbf{c}) = 0,$$

we have $\mathbf{b} \in \mathcal{N}(\mathbf{a})$. Thus, we obtain

$$(9.5) \quad h_{\omega'} \circ h_{\omega}(\mathcal{N}(\mathbf{a})) \subset \mathcal{N}(\mathbf{a}),$$

and we are done. \square

9.3. ω preserves \mathcal{PMF} . This section is devoted to show (1) in Theorem 3. Namely, we prove the following.

Proposition 9.3 (ω preserves \mathcal{PMF}). *Let $\omega: \mathcal{T}_{g,m} \rightarrow \mathcal{T}_{g,m}$ be a mapping of bounded distortion for triangles with continuous extension to $\text{cl}_{GM}(\mathcal{T}_{g,m})$. Suppose that ω has a quasi-inverse ω' which also extends continuously to $\text{cl}_{GM}(\mathcal{T}_{g,m})$. Then, the restriction of ω to \mathcal{PMF} is a self-homeomorphism of \mathcal{PMF} . Furthermore, $\omega' = \omega^{-1}$ on \mathcal{PMF} .*

The proof of Proposition 9.3 will be given in §9.3.2. In the next section, before showing Proposition 9.3, we deal with uniquely ergodic measured foliations as elements in \mathcal{C}_{GM} .

9.3.1. Uniquely ergodic measured foliations. In this paper, $G \in \mathcal{MF} - \{0\}$ is said to be *uniquely ergodic* if every $F \in (\mathcal{N}(G) - \{0\}) \cap \mathcal{MF}$ is projectively equivalent to G . It is known that the set of uniquely ergodic measured foliations are dense in \mathcal{MF} . In fact, such measured foliations consists of a full-measure set in \mathcal{MF} (cf. [27] and [42]).

In the Gardiner-Masur boundary, simple closed curves and uniquely ergodic measured foliations are rigid in the following sense.

Lemma 9.2 (Theorem 3 of [34]). *Let $p \in \text{cl}_{GM}(\mathcal{T}_{g,m})$. Let $G \in \mathcal{MF}$ be a simple closed curve or a uniquely ergodic measured foliation. Suppose that $\mathcal{E}_p(F) = 0$ for $F \in \mathcal{N}(G) \cap \mathcal{MF}$. Then there is a $t > 0$ such that*

$$\mathcal{E}_p(F) = t i(F, G)$$

for all $F \in \mathcal{MF}$. Namely, $p = [G]$ as points in $\text{cl}_{GM}(\mathcal{T}_{g,m})$.

Here, we give a characterization of uniquely ergodic measured foliations as follows.

Lemma 9.3 (Uniquely ergodic points). *The following four conditions are equivalent for $\mathbf{a} \in \mathcal{C}_{GM} - \{0\}$:*

- (i) *There exists $\mathbf{b} \in \mathcal{C}_{GM}$ such that $\mathcal{N}(\mathbf{a}) = \{t\mathbf{b} \mid t \geq 0\}$.*
- (ii) *$\mathcal{N}(\mathbf{a}) = \{t\mathbf{a} \mid t \geq 0\}$.*
- (iii) *$\mathbf{a} \in \mathcal{MF}$ and \mathbf{a} is uniquely ergodic.*
- (iv) *$\mathcal{N}(\mathbf{a})$ contains a uniquely ergodic measured foliation.*

Proof. **(i) is equivalent to (ii).** Clearly (ii) implies (i). Since $\mathcal{N}(\mathbf{a}) \neq \{0\}$, $\mathbf{a} \in \tilde{\partial}_{GM}$. Thus, (ii) follows from (i) since $i(\mathbf{a}, \mathbf{a}) = 0$ (cf. Theorem 5).

(ii) implies (iii). By (1) and (3) of Proposition 9.1, $\mathbf{a} \in \tilde{\partial}_{GM}$ and $\mathcal{N}(\mathbf{a}) \cap \mathcal{MF} \neq \{0\}$. Therefore, we have $\mathbf{a} \in \mathcal{MF}$. Thus, if $F \in \mathcal{MF}$ satisfies $I(F, \mathbf{a}) = 0$, F is

projectively equivalent to \mathfrak{a} . This means that \mathfrak{a} is a uniquely ergodic measured foliation.

(iii) implies (ii). Let $G \in \mathcal{MF} \subset \mathcal{C}_{GM}$ be a uniquely ergodic measured foliation. Let $\mathfrak{b} \in \mathcal{N}(G) - \{0\}$. From Proposition 9.1, $\mathfrak{b} \in \tilde{\partial}_{GM}$. Let $p \in \partial_{GM}\mathcal{T}_{g,m}$ and $t > 0$ with $\mathfrak{b} = \tilde{\Psi}_{x_0}(tp)$. Then, by Theorem 5,

$$t\mathcal{E}_p(G) = i(\tilde{\Psi}_{x_0}(tp), G) = i(\mathfrak{b}, G) = 0.$$

Hence, by Lemma 9.2, \mathfrak{b} is projectively equivalent to G . This means that $\mathcal{N}(G) = \{tG \mid t \geq 0\}$.

(iii) is equivalent to (iv). Clearly (iii) implies (iv) since $\mathfrak{a} \in \mathcal{N}(\mathfrak{a})$. Suppose $\mathcal{N}(\mathfrak{a})$ contains a uniquely ergodic measured foliation G . Since $i(\mathfrak{a}, G) = 0$, by applying the same argument in “(iii) implies (ii)” above, we get \mathfrak{a} is projectively equivalent to G and \mathfrak{a} is a uniquely ergodic measured foliation. \square \square

9.3.2. Proof of Proposition 9.3. Let $G \in \mathcal{MF} \subset \mathcal{C}_{GM}$ be a uniquely ergodic measured foliation. Since $\mathcal{N}(G) = \{tG \mid t \geq 0\}$, we have from Proposition 9.2 that

$$h_{\omega'} \circ h_{\omega}(\mathcal{N}(G)) \subset \mathcal{N}(G) = \{tG \mid t \geq 0\}.$$

Since $h_{\omega'} \circ h_{\omega}(G) \in h_{\omega'} \circ h_{\omega}(\mathcal{N}(G))$, $h_{\omega'} \circ h_{\omega}(\mathcal{N}(G)) \neq \{0\}$. Therefore,

$$h_{\omega'} \circ h_{\omega}(\mathcal{N}(G)) = \mathcal{N}(G) = \{tG \mid t \geq 0\}.$$

This implies that $\omega' \circ \omega([G]) = [G]$.

Since the set \mathcal{PMF}^{UE} of uniquely ergodic measured foliations are dense in \mathcal{PMF} and ω and ω' are continuous, we conclude that $\omega' \circ \omega$ is the identity mapping on \mathcal{PMF} . By applying the same argument, we deduce that $\omega \circ \omega'$ is also the identity on \mathcal{PMF} . In particular, since

$$\mathcal{PMF} = \omega \circ \omega'(\mathcal{PMF}) \subset \omega(\partial_{GM}\mathcal{T}_{g,m}),$$

\mathcal{MF} is contained in both $h_{\omega}(\tilde{\partial}_{GM})$ and $h_{\omega'}(\tilde{\partial}_{GM})$.

Let $[G] \in \mathcal{PMF}^{UE}$ again. By Proposition 9.1, we can take $F \in \mathcal{N}(h_{\omega}(G)) \cap \mathcal{MF}$ with $F \neq 0$. Since $\mathcal{MF} \subset h_{\omega}(\tilde{\partial}_{GM})$, there is an $\mathfrak{a} \in \tilde{\partial}_{GM}$ such that $F = h_{\omega}(\mathfrak{a})$. Since $i(h_{\omega}(\mathfrak{a}), h_{\omega}(G)) = i(F, h_{\omega}(G)) = 0$, we have from Proposition 9.2 that $i(\mathfrak{a}, G) = 0$. Hence, it follows from Lemma 9.3 that $\mathfrak{a} = tG$ for some $t > 0$. Therefore $h_{\omega}(G) = t^{-1}F \in \mathcal{MF}$, and $\omega([G]) \in \mathcal{PMF}$ for all $[G] \in \mathcal{PMF}^{UE}$. By applying the same argument to $h_{\omega'}$, we conclude that $\omega(\mathcal{PMF}) \subset \mathcal{PMF}$ and $\omega'(\mathcal{PMF}) \subset \mathcal{PMF}$ from the density of uniquely ergodic measured foliations in \mathcal{MF} .

On the other hand, since $\omega \circ \omega'$ and $\omega' \circ \omega$ are the identity on \mathcal{PMF} , we deduce

$$\mathcal{PMF} = \omega \circ \omega'(\mathcal{PMF}) \subset \omega(\mathcal{PMF}) \subset \mathcal{PMF}$$

and we are done. \square

9.3.3. Null space in \mathcal{MF} . From Proposition 9.3, we have the following observation.

Proposition 9.4. *Let ω be as Proposition 9.3. For $G \in \mathcal{MF} \subset \mathcal{C}_{GM}$,*

$$h_{\omega}(\mathcal{N}(G) \cap \mathcal{MF}) = \mathcal{N}(h_{\omega}(G)) \cap \mathcal{MF}.$$

Proof. Take a quasi-inverse ω' of ω . Notice from Proposition 9.3 that $\omega' = \omega^{-1}$ on \mathcal{PMF} . Therefore, the restrictions of h_ω and $h_{\omega'}$ to \mathcal{MF} are self-homeomorphisms of \mathcal{MF} and $h_\omega = h_{\omega'}^{-1}$ on \mathcal{MF} .

Take $F \in \mathcal{N}(h_\omega(G)) \cap \mathcal{MF}$. Since $i(h_\omega \circ h_{\omega'}(F), h_\omega(G)) = i(F, h_\omega(G)) = 0$, we have $i(h_{\omega'}(F), G) = 0$ and $h_{\omega'}(F) \in \mathcal{N}(G) \cap \mathcal{MF}$ from Proposition 9.2. Therefore,

$$\begin{aligned} \mathcal{N}(h_\omega(G)) \cap \mathcal{MF} &\subset h_{\omega'}^{-1}(\mathcal{N}(G) \cap \mathcal{MF}) \\ &\subset h_{\omega'}^{-1}(\mathcal{N}(G)) \cap h_{\omega'}^{-1}(\mathcal{MF}) = h_\omega(\mathcal{N}(G)) \cap \mathcal{MF}. \end{aligned}$$

Conversely, let $F \in h_\omega(\mathcal{N}(G)) \cap \mathcal{MF}$. Take $H \in \mathcal{N}(G)$ with $h_\omega(H) = F$. By Proposition 9.2 again, $i(h_{\omega'}(F), G) = i(H, G) = 0$ implies $i(F, h_\omega(G)) = 0$. Therefore, we obtain $F \in \mathcal{N}(h_\omega(G)) \cap \mathcal{MF}$ and

$$h_\omega(\mathcal{N}(G)) \cap \mathcal{MF} \subset \mathcal{N}(h_\omega(G)) \cap \mathcal{MF},$$

and we are done. \square \square

9.4. Proof of Theorem 3. From Proposition 9.3, it suffices to check the assertion (2) in the theorem.

Let $\alpha = 1 \cdot \alpha \in \mathbb{R}_+ \otimes \mathcal{S}$. We identify α as an element of $\tilde{\partial}_{GM}$ by (2.2). Then, by Proposition 9.3, $h_\omega(\alpha) \in \mathcal{MF}$. Notice that $\mathcal{N}(\alpha) \cap \mathcal{MF}$ is a subset of codimension one in \mathcal{MF} . Therefore, by Proposition 9.4, so is $\mathcal{N}(h_\omega(\alpha)) \cap \mathcal{MF}$ since h_ω is a self-homeomorphism of \mathcal{MF} . Since the complex dimension of $\mathcal{T}_{g,m}$ is at least 2, by virtue of Theorem 4.1 in [18], we deduce that $h_\omega(\alpha) \in \mathbb{R}_+ \otimes \mathcal{S}$. By applying the same argument to the quasi-inverse ω' , we conclude that the action of ω on \mathcal{PMF} preserves \mathcal{S} . Namely, ω is a bijection from \mathcal{S} onto \mathcal{S} .

Let $\alpha, \beta \in \mathcal{S}$ with $i(\alpha, \beta) = 0$. Then, $\beta \in \mathcal{N}(\alpha) \cap \mathcal{MF}$. By the argument above, $h_\omega(\beta) \in \mathcal{N}(h_\omega(\alpha)) \cap \mathcal{MF}$ and hence $i(h_\omega(\alpha), h_\omega(\beta)) = 0$. This means that $\omega: \mathcal{S} \rightarrow \mathcal{S}$ induces an automorphism of the complex of curves of X . \square

9.5. Proof of Corollary 2. The purpose of this section is to prove Corollary 2 by applying Theorem 3. Here we notice that any isometry of $(\mathcal{T}_{g,m}, d_T)$ extends to $\partial_{GM}\mathcal{T}_{g,m}$ as a homeomorphism, since $\text{cl}_{GM}(\mathcal{T}_{g,m})$ is canonically identified with the horofunction compactification of $(\mathcal{T}_{g,m}, d_T)$ (cf. [22]. See also [6], [14] and [38]).

9.5.1. Action of the extended mapping class group. Before proving Corollary 2, we shall recall the action of the extended mapping class group on Teichmüller space (cf. [16] and [30]).

The *extend mapping class group* $\text{Mod}^*(X)$ is defined by

$$\text{Mod}^*(X) = \text{Diff}(X)/\text{Diff}_0(X)$$

where $\text{Diff}(X)$ is the group of diffeomorphisms of X and $\text{Diff}_0(X)$ is a normal subgroup of $\text{Diff}(X)$ consisting of diffeomorphisms which are isotopic to the identity. Here, we may choose X so that it admits an antiholomorphic reflection $j_X: X \rightarrow X$.

Let $\psi \in \text{Diff}(X)$. If ψ is represented by an orientation preserving diffeomorphism, the action of ψ is defined by

$$\psi_*(Y, f) = (Y, f \circ \psi^{-1}).$$

If ψ is represented by an orientation reversing diffeomorphism, there is an orientation preserving diffeomorphism ϑ_ψ such that ψ is isotopic to $\vartheta_\psi \circ j_X$. Then, the action of ψ is defined by

$$\psi_*(Y, f) = (Y^*, \bar{r}_Y \circ f \circ j_X \circ \vartheta_\psi^{-1}),$$

where Y^* is the conjugate Riemann surface to Y , that is, the coordinate charts of Y^* are those of Y followed by complex conjugations, and $\bar{\tau}_Y : Y \rightarrow Y^*$ is the anticonformal mapping induced by the identity mapping on the underlying surface of Y .

The following is well-known. However, we give a proof here because the author cannot find a suitable reference in the case of the action of orientation reversing diffeomorphisms.

Lemma 9.4 (Isometry). *Any element in the extended mapping class group acts isometrically on $(\mathcal{T}_{g,m}, d_T)$.*

Proof. Let $\psi \in \text{Mod}^*(X)$. If ψ is represented by an orientation preserving diffeomorphism, the assertion is well-known (cf. e.g [16]).

Suppose that ψ is represented by an orientation reversing diffeomorphism. Let ϑ_ψ as above. From the original definition of the Teichmüller distance (2.1), we have

$$d_T(\psi_*(Y_1, f_1), \psi_*(Y_2, f_2)) = \frac{1}{2} \log \inf_{h'} K(h')$$

where h' which runs over all quasiconformal mapping from Y_1^* to Y_2^* homotopic to

$$(f_2 \circ j_X \circ \vartheta_\psi) \circ (f_1 \circ j_X \circ \vartheta_\psi)^{-1} = \bar{\tau}_{Y_2} \circ f_2 \circ f_1^{-1} \circ \bar{\tau}_{Y_1}^{-1}.$$

Since each $\bar{\tau}_{Y_i}$ are anticonformal, the action of ψ_* is an isometric. $\square \quad \square$

In the proof of the following lemma, we should remark that for any simple closed curve α on a Riemann surface Y ,

$$(9.6) \quad \text{Ext}_{Y^*}(\bar{\tau}_Y(\alpha)) = \text{Ext}_Y(\alpha).$$

Indeed, notice that any conformal metric $\rho = \rho(z)|dz|$ on Y is naturally realized as a conformal metric on Y^* by push-forwarding by $\bar{\tau}_Y$. Hence the equality holds since the extremal length is defined by the supremum of the square of ρ -length over the ρ -area, where ρ runs over all conformal metrics (cf. [1]).

Lemma 9.5 (Action at the boundary). *For $\psi \in \text{Mod}^*(X)$, the restriction of the action of ψ to $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}_{g,m}$ coincides with the canonical action of ψ on \mathcal{PMF} , that is, the continuous extension of the action $\mathcal{S} \ni \alpha \mapsto \psi(\alpha) \in \mathcal{S}$.*

Proof. Let $\psi \in \text{Mod}^*(X)$. We only check the case where ψ corresponds to an orientation reversing diffeomorphism. The other case can be treated in a similar way (cf. e.g. Theorem 1.3 of [33]).

For $\alpha \in \mathcal{S}$, we denote by $R_{\alpha,y} : [0, \infty) \rightarrow \mathcal{T}_{g,m}$ the Teichmüller geodesic ray which emanates from y and is defined by the Jenkins-Strebel differential on y whose vertical foliation is α . Let $(X_t, f_t) = R_{\alpha, x_0}(t)$ for $t \geq 0$. Let $p_\infty \in \partial_{GM}\mathcal{T}_{g,m}$ be the limit of the Teichmüller geodesic ray $t \mapsto \psi_*(R_{\alpha, x_0}(t))$.

Take $\beta \in \mathcal{S}$ with $i(\alpha, \beta) = 0$. From the proof of Theorem 5.1 of [13],

$$\text{Ext}_{X_t}(f_t(\beta)) = \text{Ext}_{R_{\alpha, x_0}(t)}(\beta) = O(1)$$

as $t \rightarrow \infty$ (see also [19]). Take ϑ_ψ as above. Since $\vartheta_\psi \circ j_X$ is isotopic to ψ ,

$$(9.7) \quad \begin{aligned} \text{Ext}_{\psi_*(R_{\alpha, x_0}(t))}(\psi(\beta)) &= \text{Ext}_{X_t^*}(\bar{\tau}_{X_t} \circ f_t \circ j_X \circ \vartheta_\psi^{-1}(\psi(\beta))) \\ &= \text{Ext}_{X_t^*}(\bar{\tau}_{X_t} \circ f_t(\beta)) = \text{Ext}_{X_t}(f_t(\beta)) = O(1) \end{aligned}$$

as $t \rightarrow \infty$ (cf. (9.6)). This means that the corresponding function \mathcal{E}_{p_∞} at the limit p_∞ satisfies

$$\begin{aligned}\mathcal{E}_{p_\infty}(\beta') &= \lim_{t \rightarrow \infty} \mathcal{E}_{\psi_*(R_{\alpha, x_0}(t))}(\beta') \\ &= \lim_{t \rightarrow \infty} e^{-d_T(x_0, \psi_*(R_{\alpha, x_0}(t)))} \cdot \text{Ext}_{\psi_*(R_{\alpha, x_0}(t))}(\beta')^{1/2} = 0\end{aligned}$$

for all $\beta' \in \mathcal{S}$ with $i(\psi(\alpha), \beta') = 0$. Since the set $\{t\beta' \in \mathbb{R}_+ \otimes \mathcal{S} \mid i(\psi(\alpha), \beta') = 0\}$ is dense in $\mathcal{N}(\psi(\alpha)) \cap \mathcal{MF}$, by Lemma 9.2, the limit p_∞ is equal to the projective class of $\psi(\alpha)$. \square \square

9.5.2. Proof of Corollary 2. Let ω be an isometry of $\mathcal{T}_{g,m}$. Then, ω extends homeomorphically to $\text{cl}_{GM}(\mathcal{T}_{g,m})$. We denote by the same symbol ω the extension. By Theorem 3 and Theorem by Ivanov, Korkmaz and Luo, there is a diffeomorphism h on X which induces the action of the complex of curves above (cf. [17], [20] and [23]). By Lemma 9.4 h acts on $\mathcal{T}_{g,m}$ isometrically and the action extends to $\text{cl}_{GM}(\mathcal{T}_{g,m})$. We denote by h_* the action of h to $\text{cl}_{GM}(\mathcal{T}_{g,m})$. Let $\bar{\omega} = \omega \circ h_*^{-1}$. By Lemma 9.5, $\bar{\omega}$ acts on $\mathcal{T}_{g,m}$ isometrically and coincides with the identity on $\mathcal{PMF} \subset \partial_{GM}\mathcal{T}_{g,m}$.

The argument below is similar to that of the proof of Theorem A in [18]. However, our situation is different from that in Ivanov's proof. Indeed, as we discussed in Introduction, we consider the Gardiner-Masur compactification while he discussed "exponential maps" of Teichmüller geodesic rays. Hence, for completeness, we proceed to prove the theorem.

Claim 9.1. $\bar{\omega}$ has a fixed point in $\mathcal{T}_{g,m}$.

Proof. Take $\alpha, \beta \in \mathcal{S}$ which fill up X . Consider a holomorphic quadratic differential q whose horizontal and vertical foliations are α and β respectively (cf. [15]). Consider the Teichmüller disk $\varphi : \mathbb{D} \rightarrow \mathcal{T}_{g,m}$ corresponding to the quadratic differential q . It is well-known that the Teichmüller disk φ is invariant under the action of a pseudo-Anosov mapping $\tau_\alpha \circ \tau_\beta^{-1}$ where τ_α and τ_β are Dehn-twists along α and β , respectively (cf. [40]). Let μ_1 and μ_2 be the stable and unstable foliations of the pseudo-Anosov mapping. For simplifying of the notation, we set $\{\lambda_i\}_{i=1}^4 = \{\alpha, \beta, \mu_1, \mu_2\}$, where the equality holds as unordered sets. Let $\theta_i \in \partial\mathbb{D}$ be the corresponding point to λ_i via φ . This means that the radial ray of direction θ_i terminates at the projective class of $\lambda_i \in \partial_{GM}\mathcal{T}_{g,m}$ (cf. [34]. See also Theorem 5.1 of [13] and Lemma 9.2). We may assume that θ_i lies on $\partial\mathbb{D}$ counterclockwise. For $i = 1, 2$, let g_i be the hyperbolic geodesic connecting θ_i and θ_{i+2} in \mathbb{D} . Then, g_1 and g_2 intersect transversely in \mathbb{D} , and $\varphi(g_1) \cap \varphi(g_2)$ consists of one point, say $x_1 \in \mathcal{T}_{g,m}$ since φ is injective.

Since each end of g_i are asymptotically tangent to the radial ray at $\partial\mathbb{D}$, $\varphi(g_i)$ is Teichmüller geodesic which terminates at the projective classes of λ_i and λ_{i+2} in the Gardiner-Masur compactification (cf. [22] and Proposition 4.9 in [38]). Notice from Theorem 1.1 in [34] that the limits of two different Teichmüller rays emanating from x_1 are different in the Gardiner-Masur compactification. Hence, the horizontal and vertical foliations of corresponding quadratic differential q_i should be λ_i and λ_{i+2} for $i = 1, 2$.

Since $\bar{\omega}$ is the identity on \mathcal{PMF} , $\bar{\omega}(\varphi(g_i))$ is also a Teichmüller geodesic terminating at the projective classes of λ_i and λ_{i+2} . By applying Theorem 1.1 in [34] as

above, we deduce that $\bar{\omega}(\varphi(g_i))$ is the Teichmüller geodesic of the holomorphic quadratic differential whose horizontal and vertical foliations are λ_i and λ_{i+2} . Thus, by Theorem 5.1 in [13], $\bar{\omega}(\varphi(g_i)) = \varphi(g_i)$ for $i = 1, 2$ and hence $\bar{\omega}$ fixes the intersecting point x_1 . \square \square

Claim 9.2. $\bar{\omega}$ is the identity on $\mathcal{T}_{g,m}$.

Proof. The discussion given here is the same as that by Ivanov in [18]. However, we include a proof for completeness.

As in the previous section, for $\alpha \in \mathcal{S}$, we denote by $R_{\alpha, x_1}: [0, \infty) \rightarrow \mathcal{T}_{g,m}$ the Teichmüller geodesic ray which emanates from x_1 and is defined by the Jenkins-Strebel differential on x_1 whose vertical foliation is α . Hence, from Theorem 1.1 in [34] again, we have that R_{α, x_1} is the only geodesic ray which emanates from x_1 and terminates at $[\alpha] \in \mathcal{PMF} \subset \partial_{GM} \mathcal{T}_{g,m}$ since $\lim_{t \rightarrow \infty} R_{\alpha, x_1}(t) = [\alpha]$ by Theorem 5.1 of [13]. Since $\bar{\omega}([\alpha]) = [\alpha]$, we deduce that $\bar{\omega} \circ R_{\alpha, x_1} = R_{\alpha, x_1}$ on $[0, \infty)$. Since Teichmüller rays $\{R_{\alpha, x_1}\}_{\alpha \in \mathcal{S}}$ are dense in $\mathcal{T}_{g,m}$, we conclude that $\bar{\omega}$ is the identity on $\mathcal{T}_{g,m}$. \square \square

For closing the proof of Corollary 2, we check that the extended mapping class group $\text{Mod}^*(X)$ is isomorphic to the isometry group $\text{Isom}(\mathcal{T}_{g,m}, d_T)$ of $(\mathcal{T}_{g,m}, d_T)$. From Lemma 9.4, there is a natural homomorphism

$$(9.8) \quad \text{Mod}^*(X) \ni h \mapsto h_* \in \text{Isom}(\mathcal{T}_{g,m}, d_T).$$

From Claim 9.2, the homomorphism (9.8) is surjective. Let $h \in \text{Mod}^*(X)$ and assume that $h_* = \text{id}$ on $\mathcal{T}_{g,m}$. Then, from Lemma 9.5, the extension of h_* to $\partial_{GM} \mathcal{T}_{g,m}$ fixes \mathcal{S} pointwise. From Theorem 3, h_* induces the identity automorphism of the complex of curves. Hence, by Ivanov-Korkmaz-Luo's theorem, h should be the identity from the assumption of X . \square

10. HYPERBOLOID MODEL

We close this paper with a hyperboloid model of Teichmüller space of the Teichmüller distance by confirming (1.7) and (1.8) which are discussed in §1.3.4.

Proposition 10.1 (Hyperboloid model). *The image of $\tilde{\Phi}_{GM}$ coincides with*

$$\{\mathbf{a} \in \mathcal{C}_{GM} \mid i(\mathbf{a}, \mathbf{a}) = 1\}.$$

Furthermore, we have the equality

$$(10.1) \quad \tilde{\partial}_{GM} = \partial \mathcal{C}_{GM} = \{\mathbf{a} \in \mathcal{C}_{GM} \mid i(\mathbf{a}, \mathbf{a}) = 0\}.$$

Proof. We first check (10.1). The equality $\tilde{\partial}_{GM} = \partial \mathcal{C}_{GM}$ follows from the definition. Let $\mathbf{a} \in \tilde{\partial}_{GM}$. By definition, $\mathbf{a} = t \Psi_{x_0}(p)$ for some $p \in \partial_{GM} \mathcal{T}_{g,m}$ and $t \geq 0$. Take a sequence $\{y_n\}_{n=1}^\infty$ in $\mathcal{T}_{g,m}$ with $y_n \rightarrow p$ in $\text{cl}_{GM}(\mathcal{T}_{g,m})$. From (5) of Theorem 5, we have

$$i(\mathbf{a}, \mathbf{a}) = \lim_{n \rightarrow \infty} t^2 i(\Psi_{x_0}(y_n), \Psi_{x_0}(y_n)) = \lim_{n \rightarrow \infty} t^2 \exp(-d_T(x_0, y_n)) = 0,$$

and hence

$$\tilde{\partial}_{GM} \subset \{\mathbf{a} \in \mathcal{C}_{GM} \mid i(\mathbf{a}, \mathbf{a}) = 0\}.$$

Conversely, let $\mathbf{a} \in \mathcal{C}_{GM}$ with $i(\mathbf{a}, \mathbf{a}) = 0$. If $\mathbf{a} \in \mathcal{T}_{GM}$, there is $y \in \mathcal{T}_{g,m}$ and $t > 0$ such that $\mathbf{a} = t \Psi_{x_0}(y)$. Therefore,

$$i(\mathbf{a}, \mathbf{a}) = t^2 i(\Psi_{x_0}(y), \Psi_{x_0}(y)) = t^2 \exp(-d_T(x_0, y)) > 0,$$

which is a contradiction. Hence, we get (10.1).

From (1.5) and (4) of Theorem 5,

$$\begin{aligned} i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(z)) &= \exp(d_T(x_0, y)) \cdot \exp(d_T(x_0, z)) \cdot i(\Psi_{x_0}(y), \Psi_{x_0}(z)) \\ &= \exp(d_T(y, z)) \end{aligned}$$

For $y, z \in T(X)$. In particular, we have

$$\tilde{\Phi}_{GM}(\mathcal{T}_{g,m}) \subset \{\mathfrak{a} \in \mathcal{C}_{GM} \mid i(\mathfrak{a}, \mathfrak{a}) = 1\}.$$

Take $\mathfrak{a} \in \mathcal{C}_{GM}$ with $i(\mathfrak{a}, \mathfrak{a}) = 1$. From (10.1), $\mathfrak{a} \in \mathcal{T}_{GM}$. Therefore, $\mathfrak{a} = t \tilde{\Phi}_{GM}(y)$ for some $t > 0$ and $y \in \mathcal{T}_{g,m}$. On the other hand, since

$$1 = i(\mathfrak{a}, \mathfrak{a}) = t^2 i(\tilde{\Phi}_{GM}(y), \tilde{\Phi}_{GM}(y)) = t^2 \exp(d_T(y, y)) = t^2.$$

Therefore, we have $t = 1$ and

$$\{\mathfrak{a} \in \mathcal{C}_{GM} \mid i(\mathfrak{a}, \mathfrak{a}) = 1\} \subset \tilde{\Phi}_{GM}(\mathcal{T}_{g,m}),$$

which is what we wanted. \square \square

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, MACHIKANEYAMA
1-1, TOYONAKA, OSAKA, 560-0043, JAPAN
E-mail address: `miyachi@math.sci.osaka-u.ac.jp`